ABSTRACT. This paper re-evaluates the problem of measuring the \textit{a priori} relative voting power of a voter in an assembly. We propose several new intuitively compelling postulates that any reasonable index of voting power ought to satisfy. At the same time we argue that most of the paradoxes of voting power discussed in the literature are paradoxical only in a weak sense, if at all. This leaves three crippling paradoxes—the well-known paradox of \textit{weighted voting}, and two new ones presented here: the \textit{bloc} and \textit{donation} paradoxes. We evaluate the four main relative power indices discussed in the literature with respect to these three severe paradoxes. The Shapley–Shubik index is seen to be immune to all three paradoxes, while the Deegan–Packel index is vulnerable to all three. The Banzhaf and the Johnston indices are demonstrably immune to the paradox of weighted voting. However, they are shown to suffer from both the bloc and the donation paradoxes. We argue that this seriously undermines these indices in a hitherto unsuspected way. Several other theoretical issues relating to voting power are discussed.

\textit{Keywords:} Index of voting power, paradoxes of voting power, simple voting game, weighted voting game.

1. INTRODUCTION

The problem we address in this paper is whether the \textit{a priori} relative voting power of a voter in a decision-making assembly can be measured; and if so, how. Although a good number of technically ingenious results have been obtained, it seems to us that from a conceptual point of view the present state of the subject is rather unsatisfactory.

As far as we know, six voting-power indices have been proposed in the literature: one by Shapley and Shubik (1954), one by Banzhaf (1965), one by Deegan and Packel (1978), one by Johnston (1978) and two by Coleman (1971). However, the Coleman indices were shown by Brams and Affuso (1976) to be proportional to the Banzhaf index (and to each other), and therefore need not concern us here. We are thus left with four indices, which are clearly independent. For brevity, we shall refer to them as ‘S-S’, ‘Bz’, ‘D-P’ and ‘Jn’, respectively.

We have not found an axiomatic characterization of the Jn index in the literature, but each of the remaining indices can be characterized axiomatically as the unique index satisfying a small number of desiderata or axioms, which are of course somewhat different in each of the three cases. In the case of the Bz index the characterization applies not to its original relative form but to the so-called \textit{absolute} version proposed by Dubey and Shapley. For details of these characterizations cf. Dubey (1975); Dubey and Shapely (1979); Straffin (1982); Deegan and Packel (1978), (1982); Owen (1978a), (1978b).
The axiomatic characterizations of the S-S and Bz indices are mathematically elegant, but both of them postulate Dubey’s (1975) Composition Axiom (cf. Straffin, 1982 p. 294), which is not intuitively compelling. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are the respective sets of winning coalitions in any two voting games with the same set of voters, then the Composition Axiom requires that the sum of any voter’s powers in $\mathcal{W}_1$ and $\mathcal{W}_2$ be equal to the sum of that voter’s powers in the two games whose sets of winning coalitions are $\mathcal{W}_1 \cap \mathcal{W}_2$ and $\mathcal{W}_1 \cup \mathcal{W}_2$, respectively.

We share the view of several authoritative commentators that the heuristic justification for this axiom is far from being overwhelming. For example, Straffin (1982, p. 296), concurring with Roth (1977), states that “if we try to think of how to interpret [the axiom] as a statement about power in political situations then the best we can say is that it appears ‘somewhat opaque’.” It seems to us that this axiom was introduced on a posteriori grounds, in order to guarantee the uniqueness of the result, rather than on grounds of a priori plausibility.

In the case of the D-P index, the axiomatic characterization supplied by Deegan and Packel (1978, p. 116) is a more or less transparent reformulation of the original explicit definition, and therefore does not afford any independent heuristic justification for this index.

Thus, as far as we know, the literature does not contain any characterization of an index of voting power by means of an intuitively compelling set of axioms. There is no shortage of individual axioms that are intuitively very convincing: several such axioms have been considered, and we shall propose some new ones. However, it seems that so far no-one has provided a complete set of such axioms, sufficient for characterizing uniquely a particular index of voting power.

Closely related to the issue of intuitively compelling postulates is that of the so-called paradoxes of voting power. A paradox is a proposition that is or appears to be absurd or contrary to common sense, but on further examination turns out to be true. Paradoxicality is a matter of degree: a true proposition may be mildly surprising or barely believable. It is also a matter of opinion: a well-informed, experienced and cautious observer is less easily surprised than a naive novice.

The negation of a mildly paradoxical proposition is a proposition that is at least apparently plausible. The negation of a paradox, in the strong sense of this term, is an intuitively compelling proposition. Thus, to say that a given index of voting power suffers from a paradox in the strongest sense is tantamount to saying that that index fails to satisfy an intuitively compelling condition, and is therefore ineligible as a reasonable measure of voting power. If it turned out that every conceivable index of voting power suffers from such crippling disqualification, this would amount to a theorem asserting the impossibility of measuring voting power by any (single) numerical index. While such an impossibility theorem cannot be ruled out in advance, nothing of the kind has been proved so far.

We shall argue that most of the voting-power paradoxes discussed in the literature – namely, the paradoxes of size, redistribution, quarrelling, and new