4. V. F. Gaposhkin (Gaposkin), "Criteria for the strong law of large numbers for some classes of second-order stationary processes and homogeneous random fields," Teor. Veroyatn. Ee Primen., 22, No. 2, 295-319 (1977).
5. V. F. Gaposhkin (Gaposkin), "A theorem on the convergence almost everywhere of a sequence of measurable functions, and its applications to sequences of stochastic integrals," Mat. Sb., 104 (16), 3-21 (1977) [MATH. USSR Sb., 33, 1 (1977)].
6. R. Duncan, "Pointwise convergence theorems for self-adjoint and unitary contractions," Ann. Probab., 5, No. 4, 622-626 (1977).
7. D. Burkholder and Y. Chen, "Iterates of conditional expectation operators," Proc. Am. Math. Soc., 12, No. 3, 490-495 (1961).
8. E. Hille and R. S. Phillips, Functional Analysis and Semigroups, Am. Math. Soc. Coll. Publ., 31 (1957).
9. V. F. Gaposhkin (Gaposkin), "The local ergodic theorem for groups of unitary operators and second-order stationary processes," Mat. Sb., 111, 249-265 (1980).

METHOD OF ORBITS IN THE REPRESENTATION THEORY OF COMPLEX LIE GROUPS

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1. Introduction

The method of orbits (see [2, 3, and 13]) originated as a method of constructing a large class of unitary representations of an arbitrary Lie group. Each representation is defined by an orbit of the group in the space dual to its Lie algebra, and it seems that one can express in terms of an orbit the properties of the corresponding representation: to calculate its character, the spectrum of the restriction to a subgroup, etc.

In the work of the French School, the algebraic version of the method of orbits is used to study enveloping algebras, their centers, and primitive ideals (i.e., the kernels of irreducible representations). In particular, for solvable Lie algebras, they have succeeded in describing the structure of the field of partial envelopings of the algebra, and finding all its primitive ideals (see [1, 4]). Similar results are proved by induction on the dimension of a Lie algebra. This method runs into difficulties when the Lie algebra is unsolvable. Sometimes these difficulties can be overcome, but the complexity of the relevant proofs increases considerably. The direct methods put forward in the present article allow us both to simplify the proofs of certain standard theorems, and to obtain new results. Our approach is intermediate between the analytic and algebraic ones. It is close to the theory of quantization, whose connection with the method of orbits was discovered by Kostant [3].

We briefly state our main results. We consider the set of generalized functions concentrated on the identity of a Lie group G. They form an algebra under convolution that is none other than the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ corresponding to the group G. If G is $\mathbf{R}^n$, then the Fourier transformation establishes an isomorphism between $U(\mathfrak{g})$ and algebras of polynomials. It turns out that if G is arbitrary, then there is a mapping (we denote it by J) of the algebra $U(\mathfrak{g})$ into the set $\mathbf{C}[\mathfrak{g}^*]$ of polynomials on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ that plays the same role as the Fourier transformation in the above example. By means of J we carry over the multiplication from $U(\mathfrak{g})$ into $\mathbf{C}[\mathfrak{g}^*]$ [i.e., we define it by the formula $\Phi \circ \Psi = J(J^{-1}\Phi * J^{-1}\Psi)$]. When $G = \mathbf{R}^n$, this operation is the usual product. In the general case it sends the space of polynomials into an algebra H isomorphic to $U(\mathfrak{g})$. By going over from $U(\mathfrak{g})$ to H we can construct a fairly large commutative subalgebra in $U(\mathfrak{g})$: the corresponding subalgebra in H consists of all polynomials constant on specified submanifolds in $\mathfrak{g}^*$.

We illustrate the construction of these submanifolds by the example of the group $G = SL(2, \mathbf{R})$. It can be verified that the orbits of the action of $G = SL(2, \mathbf{R})$ in $\mathfrak{g}^*$ that is dual to the associated action in $\mathfrak{g}$ are hyperboloids (or their components) in the three-dimensional space $\mathfrak{g}^*$. In a suitable Cartesian coordinate system they are defined by the equations $x^2 + y^2 = z^2 + c$. We consider the domain of the hyperboloids of one sheet $(c > 0)$. It is a standard fact that such a hyperboloid has a system of linear generators which can be chosen in

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two ways. For each hyperboloid we fix one of these ways so that the systems chosen on near hyperboloids are compatible. We define A as the subspace of those polynomials that are constant on each linear generator of each hyperboloid. It turns out that A is a commutative subalgebra in H, and the operation in H coincides in A with the usual multiplication of polynomials. In other words, the restriction of the mapping J^{-1}: $\mathbb{C}[\mathfrak{g}^*] \to U(\mathfrak{g})$ to A is a homomorphism of algebras.

We can choose a subalgebra A with these properties in the case of an arbitrary complex Lie group G. As a corollary we obtain the following result of Duflo (see [1, 5, and 8]): J is an isomorphism of the center of $U(\mathfrak{g})$ onto a ring of polynomials on $\mathfrak{g}^*$ invariant under G. For semisimple Lie algebras this fact was discovered by Harish-Chandra.

We consider the representation π_Ω of a Lie algebra $\mathfrak{g}$ that corresponds to an orbit Ω in general position. We extend it to a representation of the algebra $U(\mathfrak{g})$ with H (by using J); we can regard π_Ω as a representation of H. Then it turns out that the elements of the commutative subalgebra A go over into diagonal operators. More precisely, π_Ω can be realized in the space of cross sections of the bundle over Ω, and it sends a function $\Psi \in A$ into the operator of multiplication by Ψ. In particular, the center of $U(\mathfrak{g})$ goes into scalar operators: $\pi_\Omega(z) = J(z)|_\Omega$.

Next we consider π_Ω as a unitary representation of a group G. The character of π_Ω is a generalized function on G, which is closely connected with the δ-function of the orbit Ω. For orbits in general position we shall prove that tr $\pi_\Omega = J^{-1}(\delta_\Omega)$ (supporting a conjecture of Kirillov).

The material in this article is arranged as follows: the basic definitions, constructions, and theorems (with outlines of a part of the proofs) are gathered together in Sec. 2. The principal results are proved in Sec. 3 by "deforming" Lie algebras and their representations. The proofs of assertions about polarization are presented in Sec. 4.

The main results of the present article were announced in [9]. The author is glad to have this opportunity to thank A. A. Kirillov for stimulating discussions on the theory of representations.

2. Definitions and Basic Results

Let G be a connected complex Lie group, and $\mathfrak{g}$ be its Lie algebra. The space $\mathfrak{g}^*$ dual to $\mathfrak{g}$ splits into orbits under the action of G dual to the associated action. Every G-orbit is a symplectic manifold. We recall the construction of a 2-form on an orbit. In accordance with the action of G in $\mathfrak{g}^*$, to an element $x \in \mathfrak{g}$ there corresponds a vector field ξ_x (of an infinitesimal translation) touching the orbit. The value of a 2-form on the vector fields ξ_x and ξ_y at a point f is f([x, y]).

A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ that is also a maximal isotropic subspace of the form f([x, y]) (defined on $\mathfrak{g}$) is called a polarization of the functional f. It is a fact that if an orbit through f has maximal dimension (in which case f is called a regular point), then polarizations of f exist. If, e.g., $\mathfrak{g}$ is semisimple and f is a functional dual to a vector in general position in a Cartan subalgebra, then $\mathfrak{p}$ can be taken to be a Borel subalgebra.

On an orbit containing f each polarization defines a Lagrange distribution in a neighborhood of f. To determine it we must consider the image of $\mathfrak{p}$ under the mapping $x \to \xi_x(f)$ of the algebra $\mathfrak{g}$ onto the tangent space to the orbit at f, and extend the resulting subspace to other points of the orbit. It can be verified that close to f the result does not depend on the method of extension and defines a G-invariant integrable Lagrangian fibration in a neighborhood of f. We obtain (locally) a partition of an orbit into fibers. Another method of obtaining the fiber through f is to consider the orbits of f under the subgroup P corresponding to the algebra $\mathfrak{p}$.

It turns out [13] that every fiber is "plane." More precisely, let $\mathfrak{p}^\perp$ be the subspace of $\mathfrak{g}^*$ consisting of functionals that annihilate $\mathfrak{p}$. Then the following proposition holds.

Proposition 2.1. The fiber Pf is an open dense set in the linear manifold $f + \mathfrak{p}^\perp$ whose complement is an algebraic submanifold in $f + \mathfrak{p}^\perp$

Proof. First of all, $f + \mathfrak{p}^\perp$ is stable under P; therefore $Pf \subset f + \mathfrak{p}^\perp$. We prove that Pf and $f + \mathfrak{p}^\perp$ have the same dimension. For $l \in \mathfrak{g}^*$ we consider the form $B_l(x, y) = l([x, y])$, $x, y \in \mathfrak{g}$. We set $n = {}^1/_2 \operatorname{rank} B_f$; then the Lagrange manifold of Pf is n-dimensional. If the dimension of the kernel B_f is k, then $\dim \mathfrak{g} = 2n + k$, and $\dim \mathfrak{p} = n + k$, so that $\dim \mathfrak{p}^\perp = \dim \mathfrak{g} - \dim \mathfrak{p} = n$.

In fact, the fiber Pf coincides with the set $\mathcal{F}$ of all those points $l \in f + \mathfrak{p}^\perp$ for which $\operatorname{rank} B_l \geq \operatorname{rank} B_f$. For the dimension of a polarization decreases as rank B_f increases; hence $\mathfrak{p}$ is a polarization of any point of $\mathcal{F}$. Therefore, the P-orbits partition $\mathcal{F}$ into open sets. But $\mathcal{F}$ is connected since its complement is a complex