QUASIERSONANCES IN THE PROBLEM OF FORCED VIBRATIONS
OF A THIN ELASTIC SHELL INTERACTING WITH A LIQUID

D. G. Vasil'ev and V. B. Lidskii

Introduction. In the development of [1] a system of four linear equations

\[ hE_0 \left( t + i\gamma \right) L_{pq} u_q (\alpha) = h\rho_0 \omega^2 u_p (\alpha) + g_p (\alpha), \]
\[ p = 1, 2, q = 1, 2, 3, \]  
\[ \alpha \in \Gamma, \quad x \subseteq V, \]

\[ (0.1) \]

\[ hE_0 \left( t + i\gamma \right) L_{3q} u_q (\alpha) = h\rho_0 \omega^2 u_3 (\alpha) + \rho_1 \omega^2 \psi \big|_\beta + g_3 (\alpha), \]
\[ q = 1, 2, 3, \]  
\[ \alpha \in \Gamma, \quad x \subseteq V, \]

\[ (0.2) \]

\[ -c_1 \Delta \psi (x) = \omega^2 \psi (x), \]

\[ (0.3) \]

\[ \alpha \subseteq \gamma. \]

By \( \beta \) we denote the derivative with respect to the exterior normal to \( \Gamma \).

In distinction from [1], in (0.1)-(0.3) Young's modulus is taken in the form \( E = E_0 (1 + i\gamma) \) where \( \gamma > 0 \) is the magnitude that characterizes constructive friction in the shell and is small in applications. In the following we assume that

\[ h \mu \ll \gamma \ll h \mu^*, \quad 0 < \mu_2 \ll \mu_1 < 1. \]  
\[ (0.5) \]

To facilitate reading of the article, we shall indicate that operators of the shell theory have structure \( L_{pq} = (h^2/12) n_{pq} + l_{pq}, p, q = 1, 2, 3 \); differential operators \( n_{pq} \) and \( l_{pq} \) are expressed by the components of the first and second metric forms of the surface (see [3, pp. 77, 78; 4, pp. 8, 40, 41]). We shall find as well that the operator \( \| L_{pq} \|_{p, \alpha}, q=1 \) is formally self-adjoint and elliptical in the Douglass--Nirenberg sense with system of indices \( s_1 = s_2 = 1, s_3 = 2, t_1 = t_2 = 1, t_3 = 2 \). When \( h = 0 \) the resulting degenerate (momentless) operator \( \| L_{pq} \|_{p, \alpha}, q=1 \) is self-adjoint as well and in the case of positive principle curvatures it is elliptical: \( s_1^* = s_2^* = 1, s_3^* = 0, t_1^* = t_2^* = 1, t_3^* = 0 \).

As in [1], below we assume V. Z. Vlasov's simplifying assumption that \( n_{pq} = 0, p + q < 5 \). All basic results are preserved even without this assumption.

We recall that in [1] the self-adjoint operator \( A \) is connected with the left-hand sides of (0.1)-(0.3) under condition (0.4) and \( \gamma = 0 \). The lower part of the spectrum of \( A \) (of frequencies of free vibrations of the shell--liquid system) is very dense. For the partition function \( n_\mu (\lambda, \alpha) \) of points of the spectrum \( \lambda_k (\lambda = \rho_0 (1 - v^2)\omega^2 / E_0) \) when \( h \to 0 \) we have the formula \( n_\mu (\lambda, \alpha) = c_1 (\lambda) h^{-6/5} + c_2 (\lambda) h^{-4/5} + c_3 (\lambda) h^{-2/5} + o(h^{-3/5}) \). Here \( c_1 (\lambda) \) and \( c_2 (\lambda) \) are the same as in [1], and \( c_3 (\lambda) = (20\pi)^{-1} (12 \rho_0 \lambda / \rho_0)^{1/5} \int_\Gamma H_\alpha (\alpha) dS, \)

\( H(\alpha) \) is the average curvature of \( \Gamma \). The third term of the asymptotics, in agreement with [5], exists under fulfillment of the geometric condition of Duistermaat and Guillemin [6].

Calculation of the constructive friction \( \gamma \) leads to a regular perturbation of operator \( A \). The frequencies of free vibrations are displaced from the positive real semiaxis to the upper complex \( \omega \) half-plane, but since \( \gamma \) is small they remain in a small neighborhood of the real
The corresponding points are called quasiresonances of problem (0.1)-(0.4). They are poles of the meromorphic resolvent $\mathcal{R}_0$ of this problem. In [1] it is shown that the spectrum of free vibrations of a shell-liquid system falls into three series: quasitangential, quasi-transverse, and liquid frequencies. We retain these terms also for the corresponding quasiresonances.

The goal of the present article is the proof of the asymptotic expansions in powers of $h$ for the solution of problem (0.1)-(0.4) (Theorem 1 of Sec. 1). These formulas can be used for analysis of the solution and establishment of its dependence on the numerous parameters entering into the system. In particular, frequencies of external loads more dangerous for the shell-liquid system, and the role of quasiresonances in this problem, appear by means of the asymptotic formulas (Theorem 2, Sec. 4). It is shown that for smooth loads only a negligible part of the quasiresonances is substantively excited (tangential and liquid). For more detail see Sec. 4.

In conclusion we note that a large literature on mechanics is dedicated to problems similar to (0.1)-(0.4). The reader will find a bibliography in [7, pp. 371-396].

1. Formulation of the Fundamental Theorem.

Let us agree to denote the trace of function $\psi(x)$ on the boundary $\Gamma$ of region $V$ by $\theta_0 \psi$, and its derivative with respect to the exterior normal to $\Gamma$ by $\theta_1 \psi$. The condition (0.4), in this way, can be expressed in the form $\theta_1 \psi = u_3(\alpha)$ and the second term from the right in the form $\rho E_0^2 \theta_0 \psi$.

First we introduce two auxiliary assertions.

**Lemma 1.** For any $g = \text{col}(g_1(\alpha), g_2(\alpha), g_3(\alpha)) \in L_2(\Gamma)$ and real $\omega > 0$ the problem (0.1)-(0.4) has a unique solution

$$f = (u_1(\alpha), u_2(\alpha), u_3(\alpha), \psi(x)) \in H^{2,2,4}(\Gamma) \oplus H^0(V),$$

$$\beta = 11/2.$$

**Lemma 2.** Let $g$ and $f$ be the same as in Lemma 1. Put $f = \mathcal{R}_0 g$, where by $\mathcal{R}_0$ we denote the resolvent of the linear operator generated by problem (0.1)-(0.4). If we evaluate all components of $f$ in the $L_2$-metric, then

$$\|\mathcal{R}_0\|_{\omega} \leq C/(\rho h^2), \quad h \to 0,$$

with constant $C$ the same for all $\omega_{\text{min}} \leq \omega \leq \omega_{\text{max}}, \omega_{\text{min}} > 0$.

We outline a plan of the proof of Lemma 1 and at the same time introduce the notation used in the following. By analogy with [1, (1.7)] we consider a homogeneous elliptic self-adjoint operator of second order

$$B = \|L_2\|_{\omega=1},$$

and denote $B_0 = B - [\rho E_0^2/E_0(1 + i\gamma)]I$. It is obvious that for all real $\omega > 0$ and $\gamma > 0$, the operator $B_0$ is invertible. Using this, we solve Eq. (0.1) with respect to $u_1(\alpha), u_2(\alpha)$:

$$\text{col}(u_1, u_2) = -B^{-1}_0 M u_3 + [\rho E_0(1 + i\gamma)]^{-1} B^{-1}_0 \text{col}(g_1, g_2).$$

Putting $u_1$ and $u_2$ from (1.4) into (0.2), we arrive at the equation

$$h E_0 (1 + i\gamma)(L_{23} - C_0) u_2 - \rho \omega^2 u_3 - \rho_1 \omega^2 \theta_0 \psi = -M^* B_0^{-1} \text{col}(g_1, g_2) + g_3,$$

in which $C_0 = M^* B_0^{-1} M$, $M^*$ is the Hermitian conjugate to $M$; we note that $C_0$ is a bounded operator when $\gamma > 0$, since $B_0^{-1}$ is of order $-2$, and $M$ and $M^*$ are of first order. We shall show also that

$$L_{23} = \frac{h^2}{12} n_{33} + l_{23} = \frac{h^2 (\Delta^r + r_{33})}{12 (1 - \nu^2)} + \frac{R_{33}^r (\alpha) + 2v R_{33}^1 (\alpha) R_{33}^1 (\alpha) + R_{33}^2 (\alpha)}{1 - \nu^2},$$

(see [3, pp. 77, 78]). In (1.6) $\Delta^r$ is the Laplace-Beltrami operator on $\Gamma$; $R_{33}^{-1}(\alpha)$ and $R_{23}^{-1}(\alpha)$ are the principle curvatures of $\Gamma$; $r_{33}$ is a second-order operator; $0 < \nu < 1/2$ is the Poisson coefficient. Both operators $n_{33}$ and $l_{23}$ in $L_{23}$ are positive-definite. We reduce the system (1.5), (0.3), and (0.4), connecting $u_3(\alpha)$ and $\psi(x)$, to the equivalent system of two Fredholm equations. We put $\psi(x) = \varphi(x) + \chi(x)$ and first change problem (0.3), (0.4) to the following: