HIGH-FREQUENCY ASYMPTOTICS OF WAVE FIELD DIFFRACTED
BY PLANE ANGULAR SECTOR. I

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The problem of plane-wave diffraction by an angular sector is examined. It is assumed that the wave process is described by the Helmholtz equation and that the Dirichlet or Neumann boundary conditions are satisfied on the sector. An exact solution of the problem is constructed in the form of a Sommerfeld integral, which is convenient for study of the problem in a high-frequency approximation.

Problems of diffraction of scalar and vector wave fields by a plane angular sector have been studied extensively [1-7]. Interest in these problems within the scope of the high-frequency (HF) diffraction theory results from the fact that, firstly, a sector is a model example of a plane shield with a nonsmooth edge (aperture) [8].

Secondly, from the point of view of geometrical diffraction theory (GDT), diffraction by a sector is interesting in that it produces waves of various types: incident, reflected by a sector plane, edge-diffracted, spherical (diffracted by a sector vertex), and multiply edge-diffracted waves. Their quantitative description and the study of their interaction are always highly interesting.

Thirdly, a plane angular sector is a degenerate case of an elliptical cone [2], being a coordinate surface in the so-called sphericoconical coordinate system — one of 11 coordinate systems in which the Helmholtz equation permits the complete separation of variables ([11], Ch. 5). This makes it possible to construct exact solutions of a diffraction problem in the form of a series in eigenfunctions that are expressed in terms of Bessel functions and periodic Lamé functions [1, 2, 4-7]. It is not clear, however,
how to extract directly from an exact solution the HF asymptotics, which must describe diffraction processes in accordance with GDT.

We shall show that all of the effects predicted by GDT can be extracted from an exact solution that is transformed beforehand to a Sommerfeld integral with standard phase and amplitude in the form of a function on a unit sphere cut along a geodesic segment. The asymptotics of the Sommerfeld integral can be studied fully by a combination of a stationary-phase method and analysis of amplitude singularities (Part II of this article will be devoted to that topic). This makes it possible to investigate completely diffraction processes in an HF approximation. On the one hand, our constructions show that the asymptotics of the exact solution agrees with the postulates of GDT, thereby rigorously confirming the latter (in the case of a sector). On the other hand, what is no less important, we shall extract from an exact formula specific quantitative characteristics that describe diffraction processes in an HF approximation. In particular, we shall derive explicit formulas for a spherical wave diffracted by a sector vertex (Part III).

The constructions of the given work are based on a general scheme of HF diffraction by cones of arbitrary cross-section [12, 13]. We limited our examination to a scalar diffraction problem with ideal boundary conditions (Dirichlet and Neumann). However, the proposed scheme can be adapted to problems of electromagnetic-wave diffraction by an absolutely conducting sector [14]. The author intends to devote a separate article to this case.

1. CONSTRUCTION OF EXACT SOLUTION IN THE FORM OF A SOMMERFELD INTEGRAL

1.1. Formulation of Problem

Let a wave process in a three-dimensional Euclidean space be described by the Helmholtz equation

\[(\Delta + k^2)u(\vec{r}) = 0.\] (1)

Here, \(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\) is a Laplace operator and \(r = (x, y, z)\).

We shall examine as an obstacle a plane angular sector \(S\) located in the plane \(y = 0\) that has a vertex at the coordinate origin and is symmetrical with respect to the negative part of the \(z\) axis (Fig. 1). We shall assume that the sector \(S\) has an opening \(2\alpha\), where \(0 < \alpha < \pi/2\). *

The problem of diffraction of the field of a point source located at point \(r_0 = (x_0, y_0, z_0) \notin S\) by sector \(S\) is formulated as follows [2]. Let \(\Omega = \mathbb{R}^3 \setminus S\) is the exterior of sector \(S\). The boundary of region \(\Omega\) consists of two banks of sectorial cross-section \(S_+\) and \(S_-\). We shall seek a function \(G(\vec{r}, r_0) \in C(\Omega \setminus \{r_0\})\) that is continuous in the closure \((\Omega \cup S_+ \cup S_-) \setminus \{r_0\}\) and satisfies the Helmholtz equation with a delta function on the right side

\[(\Delta + k^2)G(\vec{r}, \vec{r}_0) = -\delta(\vec{r} - \vec{r}_0),\] (2)

the Dirichlet or Neumann boundary condition on the sector \(S\)

\[G|_S = 0 \quad \text{or} \quad \partial G/\partial n|_S = 0\] (3)

(it is assumed that in the case of the Neumann condition, \(\text{grad} G\) exists and is continuous at the regular points of \(S_+\) and \(S_-\)), at the radiation conditions at infinity.

*In the case of \(\alpha > \pi/2\), we have a so-called "complementary" sector [7].