FAMILIES OF RECURSIVE PREDICATES
OF MEASURE ZERO

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Our aim is to render precise the meaning of statements of the type, "Almost all recursive predicates have (do not have) a given property," and to describe a method (Theorem 2) whereby the truth of such statements can be relatively easily verified. We give an example of the use of this method to characterize a number of properties of recursive predicates formulated in terms of constraints on the complexity of their regular approximations (see [1]).

1. Let us fix the Gödel numbering discussed in [2] for partial recursive functions (PRF's). We denote the function numbered $i$ by $i$. We denote by $I$ the set of natural numbers $i$ for which $i$ is a general recursive function (GRF) assuming one of the two values 0, 1. We denote by $[i]$, where $i \in I$, a word of the form $i(0) i(1) ... i(n-1)$.

Let $x$ be a binary cortege (word in the alphabet {0, 1}); we denote by $l(x)$ the length of the cortege $x$, and by $<x>$ a natural number such that for any natural $n$ the following condition holds: If $n < l(x)$, then $i(x) \in I$ coincides with the $n$-th letter to the left in the cortege $x$, otherwise $i(x) = 0$.

We call the set of natural numbers $i$ such that $i \in I$ and $[i]$ a word of the form $i(0) i(1) ... i(n-1)$, the Baire $x$ ball and denote it by $B[x]$; we define the radius $\rho(x)$ of the ball $B[x]$ as the number $2^{-l(x)}$. We call a binary cortege $x$ a code of the Baire ball $B$ if $B = B[x]$.

Let $M$ be a set of natural numbers, $M \subset I$. By analogy with [3], we introduce the following definitions. An algorithm $\Delta$ that converts natural numbers into codes of Baire balls (binary cortegees) is called a covering of the set $M$ if for any natural number $n$ in $M$ it is possible to construct a natural number $m$ such that $m \in B[\Delta(n)]$.

We call a covering $\Delta$ regular if the sequence of numbers

$$\sum_{m \in M} \rho(\Delta(m))$$

converges constructively. We call a covering $\Delta$ $\varepsilon$-bounded (where $\varepsilon$ is a positive rational number) if


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We say that a set $M$ has measure zero in $G$ if for any positive rational number $\varepsilon$ a regular $\varepsilon$-bounded covering of $M$ is realizable.

The foregoing definition of a set of measure zero is a natural one, being analogous to the definition of the set of constructive real numbers of measure zero (investigated in [3]). In a number of situations, however, the proof of statements of the type, "The set $M$ has measure zero in $G$," which relies directly on this definition, is extremely cumbersome in its development. It proves possible to simplify such proofs by an equivalent characterization of the concept of a set of measure zero without the use of the covering concept.

We say that a covering $A$ of a set $M$ is translation-invariant if for any natural number $n$, the algorithm $A^{(n)}$ defined by the condition

$$\forall i \left( A^{(n)}(i) = A(n+i) \right)$$

is also a covering of the set $M$.

**Lemma 1.** Let $M$ be a set of natural numbers, $M \subset G$. For $M$ to have measure zero in $G$ it is necessary and sufficient that a regular translation-invariant covering of $M$ be realizable.

**Proof.** Necessity. Let $M$ have measure zero in $G$. We introduce the notation $\mathcal{E}_\kappa \equiv 2^{-\kappa}$, denote by $\Lambda_\kappa$ a regular $\mathcal{E}_\kappa$-bounded covering of $M$, and by $\Lambda$ the algorithm defined by the condition

$$\forall n \left( \Lambda(n) = \Lambda_{\mathcal{E}_n}(\nu(n)) \right),$$

in which $\ell(n)$, $\tau(n)$ are the left and right members, respectively, of a pair of natural numbers with quantifier number $n$ (see [4], p. 63). The regularity of the covering $\Lambda$ follows from the constructive convergence of a sequence of the form

$$\sum_{\mathcal{E}_n} \rho(A_\kappa(i)) \quad (\kappa=0,1,...)$$

and the constructive convergence of the sequence

$$\sum_{\mathcal{E}_n} 2^{-i}.$$

Moreover,

$$\forall m \in M \forall \kappa \exists n_{>\kappa}(m \in D(n)).$$

Therefore, for any $\kappa$ the algorithm $A^{(\kappa)}$ is a covering of $M$.

Sufficiency is obvious.

Let $\mathcal{F}$ be a recursive set of binary cortege; we introduce the notation