PROPAGATION OF SHOCK WAVES IN AN ISENTROPIC, NONVISCUS GAS

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UDC 517.948.533

A new method is considered for constructing the dynamics of motion of a shock front moving with speed close to that of sound in a nonviscous, isentropic gas. The algorithm for constructing the solution reduces to the problem of successively solving systems of ordinary differential equations which provides considerable economy as compared with the known method of difference schemes for computer calculations.

INTRODUCTION

1. Formulation of the Problem. In the present work a new method will be presented for constructing the shock front with speed not differing appreciably from the speed of sound. This method is of general character and can be applied to investigate the behavior of shock waves for a broad class of nonlinear problems. We demonstrate it here in detail for equations describing the motion of an isentropic gas in an external force field. In the acoustic limit it leads to the propagation of a discontinuity in an inhomogeneous media.

In the paper certain necessary conditions are obtained for the existence of a piecewise-smooth solution of the equations of an isentropic gas which suffers a discontinuity of a smooth surface. In order to give the reader an idea of the nature of these conditions, we present them for the linearized problem (cf. [4]). In order that there exist a solution of a linear system having a discontinuity (at time t) on a smooth surface \( \Omega_t \) it is necessary that (cf., e.g., [4]): a) this surface satisfy the characteristic equation; b) the vector of jumps of the solution at the discontinuity be a null vector of the characteristic matrix of the system.

In the linear case it is not difficult to obtain necessary conditions on the jumps of the derivatives of solutions at the discontinuity. We point out that the well-known Hugoniot conditions are the analogue of condition b) in the nonlinear case.

It is obvious that the necessary condition a) is very important, since it determines the equation of the surface of discontinuity, and, knowing the surface of discontinuity at the initial time, we can construct the surface of discontinuity at time t by solving a system of ordinary differential equations for the bicharacteristics.

Here we shall find necessary conditions for nonlinear systems which are such that they enable us to solve the analogous problem for discontinuities of nonlinear equations.

It is known that the behavior of a discontinuous solution of nonlinear systems of equations depends essentially on the form of integral conservation laws which reduce to a particular partial differential equation. We shall assume that the equation is already given in divergence form and use the well-known definition of a discontinuous (generalized) solution of such an equation (cf., e.g., [2]). This means that a nonlinear operator given in divergence form can be interpreted as an operator from a space of piecewise-smooth functions to a space of generalized functions. A piecewise-smooth function is a solution if the generalized function obtained after applying the nonlinear operator is equal to zero. Of course, this definition of a discontinuous solution coincides with the usual one based on the initial integral conservation laws.

The question of finding sufficient conditions for the existence of piecewise-smooth solutions of the general nonlinear equations of gasdynamics is itself of great current interest, but we shall not consider this here. Since we have obtained only necessary conditions for existence, all the results presented in the paper are obtained assuming that a piecewise-continuous solution of the equation exists. This assumption is equivalent to assuming the existence of shock waves corresponding to the nonlinear system of equations considered. For the case of the system of equations of motion for an isentropic, nonviscous gas which we consider here there can be no doubt as to the existence of such waves (cf. [4]).


0096-4104/80/1301-0119 $07.50 ©1980 Plenum Publishing Corporation 119
A second important circumstance of which we make essential use in the mathematical formulation of the
problem is the following. It is well known (cf. [4]) that the propagation speed of shock waves is greater than
the speed of sound, and therefore the shock wave itself does not affect the magnitude of the density and the
speed of the gas in front of the shock front. This means that we can consider the separate problem of the
evolution of the speed and density in front of the shock front, since this evolution is uniquely determined by
the values of the density and speed before the front at the initial moment. We may, e.g., extend the density
and speed of the gas at the initial time before the wave front to the remainder of space in an arbitrarily smooth
manner and show that the value of the density and speed at time \( t \) before the wave front does not depend on the
manner of extension. Thus, the problem of determining the speed and density before the shock wave is the
problem of finding a solution of the system satisfying smooth initial data which is smooth in the given region
before the front. This problem presents substantially fewer difficulties for computer calculations. Moreover,
in practical problems the flow of fluid or gas in front of a shock wave is considered to be stationary, and
hence the same as before the shock wave arises; it is, as a rule, assumed to be given. We therefore hence-
forth assume that the value of the density and speed of the gas in front of the wave front is known, and the
final formulas for the surface of discontinuity we shall express in terms of these quantities. From the as-
sumption of the existence of a piecewise-smooth solution suffering a discontinuity on some smooth surface
it is possible to obtain in a known way relations between the values of the density and speed on the jump bound-
ary—the so-called Hugoniot conditions. Proceeding from the uniqueness of the expansion in smooth asymptotic
series, we here obtain analogous relations between the first derivatives of the speed and density on the jump
boundary and then between the second derivatives on the jump boundary.

Further, a nonclosed system of coupled equations is written on the discontinuity. Here it is found that
in the entire system the coupling is realized by a term of the same type which is proportional to the ratio of
the difference between the speed of the shock wave and the sound speed to the sound speed. This ratio is called
the coupling coefficient. The situation is that if a discontinuous solution of an equation depends on some small
parameter, then the term "piecewise-smooth" solution is, of course, extended to the dependence of this solu-
ton the parameter. In other words, the surface of discontinuity should not degenerate for any value of the
parameter, and the jumps of the function and its derivatives should remain bounded as the parameter tends to
zero. This is natural in the formulation of a problem with a small parameter (cf., e.g., [3]). Therefore, in
formulating the problem of necessary conditions for the existence of a piecewise-smooth solution of an equa-
tion containing a small parameter, it is necessary to assume that there exists a piecewise-smooth solution
which is discontinuous on a smooth surface and is regular with respect to a small parameter. From this the
conclusion may be drawn that the unknown function in the coupled system by which the coupling coeffi-
cient is multiplied remains bounded as the small parameter tends to zero and hence its product with the coupling coef-
cient is small and may be neglected in first approximation. This leads to uncoupling of the system, and it is
possible to construct the series of perturbation theory. Determination of the conditions on the system for the
first and especially for the second derivatives is very complicated, and computation of even three terms in
the series of perturbation theory presents major difficulty. My students V. A. Ptitsyn and V. A. Tsupin have
been of great assistance in the computations. I wish to offer them my sincere thanks.

2. Outline of the Derivation of the Formulas. We shall now outline the basic paths for obtaining the
necessary conditions. We consider first the last step—the construction of the series of perturbation theory
in the coupling coefficient.

The system of coupled equations on the surface of discontinuity obtained below has the following form:

\[
\begin{align*}
\frac{\partial \alpha}{\partial t} + L_1(s) + \alpha(s, \rho_i) &= 0, \\
\frac{\partial \rho_i}{\partial t} + L_2(s, \rho_i) + \gamma \\n\frac{\partial \rho_i^{k-1}}{\partial t} + L_k(s, \rho_i, \rho_1, \ldots, \rho_{k-1}) + \gamma P_k &= 0, \\
&\ldots \\
\end{align*}
\]

where \( \gamma = \gamma(\rho_i) \) is the coupling coefficient, \( \alpha(s, \rho_i) \) is a smooth function such that \( \alpha = 0 \) for \( \gamma = 0 \), and \( L_1, L_2, L_3, \ldots \) are certain nonlinear differential operators of first order. The surface of discontinuity is defined by
the equation \( s = s(x, t) = 0 \). If for \( s(x, t) \) we wish to take \( k \) terms of the series of perturbation theory in \( \gamma \); \( s = s_0 + s_1 + \ldots + s_k \), then for \( \rho_i \) it obviously suffices to take only \( k - 1 \) terms of the series, \( \rho_i = \rho_{i0} + \rho_{i1} + \ldots + \rho_{i,k-1} \), and for \( \rho_i \) it suffices to take \( k - i \) terms of the series \( \rho_i = \rho_{i0} + \ldots + \rho_{i,k-1} \).

Substituting in the equation for \( s \) in place of \( \rho_i \) the quantity \( \rho_{i0} \), we find \( s_1 \). Substituting \( s_0 + s_1 \) and \( \rho_{i0} \) into
the equation for \( \rho_i \) in place of \( s \) and \( \rho_i \), respectively, we find \( \rho_{i1} \). Substituting in the equation for \( \rho_i \) in place of \( s \),

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