Remark. 1) \[ \log_2 |W| + 1 \leq C_3(w) \leq |W|, \quad 1 \leq l \leq 4 \]

2) The obvious upper bound \( t_3(n) \) of the time of operation of the natural algorithm that computes \( C_3(w) \) on words of length \( n \) can be estimated as \( O(2^{n^2}) \).

In conclusion, let us present an upper bound \( C_4 \) of the complexity \( C_3 \) that can be calculated in a time which is polynomial in \( |w| \):

\[ C_4(W) = C_4(V) + 2^4 IV,9) + 4, \]

where \( \rho_4(V,p) = \sum_{u \in V} \rho(u,p); \ w = \nu p \). The operating time \( t_4(n) \) of the algorithm of calculation of \( C_4(w) \) on words of length \( n \) can be estimated as \( t_4(n) \leq t_4(n - 1) + c \cdot n^5 \) (\( c \) does not depend on \( n \)); therefore, \( t_4(n) = O(n^6) \).

LITERATURE CITED


TWO SYSTEMS FOR PROVING TAUTOLOGIES, BASED ON THE SPLIT METHOD

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We consider two calculi in which all propositional tautologies, and only these, are provable. Despite their simple structure (each calculus has one axiom and one rule of inference), we may obtain linear estimates for the length of deduction in them for various frequently occurring classes of formulas, whose tautology is recognized in polynomial time. We study the characteristics of formulas which influence the length of deduction in these systems. In particular, one of these characteristics is the degree of symmetry defined by automorphisms of the formulas.

In this article, we describe the calculi \( S \) and \( S_I \), in which all propositional tautologies, and only these, are deducible. Each of them has one axiom and one two-premise rule of inference, whose application allows us to deduce the formula \( \varphi \) if both results of the split of \( \varphi \) by some variable are provable (a split of \( \varphi \) by \( x \) is the substitution of truth values in place of \( x \)). The calculus \( S_I \) differs from \( S \) in that its objects are obtained by "sticking together" a number of objects in \( S \). Formulas in \( S_I \) are considered to within isomorphism, i.e., renaming of variables and changing places of variables. At the expense of this, when we go from \( S \) to \( S_I \) there may be a considerable shortening of the length of deduction; we give an example where the minimal length of deduction in \( S \) depends exponentially on the number of variables, but in \( S_I \) is linear.

Despite such evidence of the boundedness of the tools of $S$, we are able to obtain linear estimates for the length of deduction in $S$ for various frequently occurring classes, where tautology is recognized in polynomial time. This refers to the class of formulas is DNF (Disjunctive Normal Form), in which conjunctions contain no more than two terms, and to classes of "Horn" formulas with locally occurring variables, and others. We establish an upper bound for formulas with conjunctions containing no more than three terms $\lambda n^3$, where $n$ is the number of variables, and $\lambda \approx 1.77$.

A characteristic of the structure of formulas which influence the length of deduction in $S$ and $S_1$ is, in particular, the following: The set of variables may be split up into subsets (blocks) such that variables in one block are "chained" together more strongly than with those in other blocks. We consider the connection between the parameters of this subdivision and the length of deduction in $S$. If the occurrence of the variables in a block has some symmetry (a symmetry is defined by automorphisms of the formulas), then we can use this property for shortening deductions in $S_1$.

The terminology relating to complexity, graphs, etc., is the same as in [1]; the concepts connected with splits are defined in Sec. 1.

1. The Tree of Splits

We introduce some definitions and notation connected with the split method (see [2]). We consider a propositional formula in disjunctive normal form as a set of conjunctions, and a conjunction as a set of literals (as literal is a variable or a variable with negation; let $\alpha$ be a literal, then $\bar{\alpha}$ denotes the contrary literal to $\alpha$). Thus, two formulas are distinct if the corresponding sets are distinct. The set of values of variables, like conjunctions, will also be represented by sets of literals; e.g., the set $(x_1 = "true"," x_2 = "false"," x_3 = "false")$ corresponds to the set $\{x_1, x_2, x_3\}$, which we write as the row $x_1x_2x_3$. We say that the conjunction $C$ is contrary to the set $\alpha_1,\ldots,\alpha_k$ if $C$ contains at least one of the literals $\alpha_1,\ldots,\alpha_k$.

Let $F$ be a formula, and $t$ a set of values of certain variables. The result of substituting the set $t$ in $F$, written $F[t]$, is defined as the formula obtained from $F$ as follows:

1) We remove from $F$ conjunctions contrary to $t$ (if all the conjunctions are removed, $F[t]$ is assumed to be the empty set $\phi$);
2) from each remaining conjunction, we remove the literals which also occur in $t$ (if all the literals are removed from some conjunction, then $F[t]$ is defined as the formula consisting of the empty conjunction and is denoted $\Box$);
3) conjunctions containing no variables from the set $t$ remain unchanged.

A counterexample to $F$ is a set $t$ such that $F[t] = \phi$, i.e., a set contrary to all conjunctions in $F$. A tautology is a formula with no counterexample (in particular, $\Box$ is a tautology). Two formulas are deductively equivalent if they are either both tautologies or both nontautologies.

We call the results of the split of $F$ by the variable $x$ the two formulas $F[x]$ and $F[\bar{x}]$, obtained from $F$ by substituting opposite values of $x$. It is easily verified that a counterexample to $F$ is also a counterexample to at least one of the formulas $F[x]$ and $F[\bar{x}]$; the converse is also true; if $t$ is some counterexample to $F[\alpha]$ (we may assume that $t$ does not contain the literal $\bar{\alpha}$), then the set $t \cup \{\alpha\}$ is a counterexample to $F$. Hence it follows that $F$ is deductively equivalent to $F[x] \& F[\bar{x}]$.

Henceforward, $F$ will denote everywhere a formula with $n$ variables, $X$ the set of its variables ($|X| = n$), and $Y$ a nonempty subset of $X$. The tree of splits (t.s.) for $F$ by $Y$ is the premise binary tree $T$, in which each node $v$ is assigned a formula, denoted by $F_v$, for which the following conditions are satisfied:

1) The root is assigned the original formula $F$;
2) formulas assigned to finite nodes do not contain variables in $Y$ (these are called finite formulas of $T$ and, in particular, may include $\Box$ and $\phi$);
3) the sons of the node $v$ are assigned formulas which are results of the split of $F_v$ by $y$, i.e., $F_v[y]$ and $F_v[\bar{y}]$, where $y$ is a variable belonging to $Y$ and occurring in $F_v$. 

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