subdivision is transitive and, therefore, by Theorem 9 its I-weight is not greater than $\frac{1}{2} \times (l + 1)(l + 2)$. Since for the formulas $O^k_l$ we have the restrictions $1 \leq k \leq l \leq m$, $I(O^m_l)$ is not greater in order than $m^2$, i.e., the I-complexity of the formula $O^m_l$ depends quadratically on the number of variables, which is equal to $m(m - 1)$.

LITERATURE CITED


SIMPLIFICATION OF GENTZEN'S REDUCTIONS IN CLASSICAL ARITHMETIC

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The reduction of proofs of classical arithmetic applied by G. Gentzen in 1936 has not so far been used. The article offers a reduction which is a considerable simplification of that reduction. It can be used for proof normalization. The proof ordinal used is strictly monotone.

In 1936, Gentzen published a proof of consistency of classical arithmetic using transfinite induction to $\varepsilon_0$ [1]. In 1939, Gentzen published the second proof of consistency [2]. While the second proof was later restated and the proof reduction offered in it found some applications, the first proof remained in its primary form and has not been used. Perhaps this can be explained by the fact that Gentzen's text uses clumsy definitions requiring some clarifications.

A considerably simpler proof reduction based on Gentzen's 1936 ideas is offered below. For the sake of simplicity, a proof reduction with a finite elementary sequent is considered but it is readily extended to an arbitrary proof, similarly to what has been done in [3] with respect to the modified reduction from [2]. As in [1], the proof ordinal used is strictly monotone, unlike the one used in [2] (see Sec. 3). The suggested reduction is used in the calculus with multiformulaic succedento.

1. The language of the studied variant of classical arithmetic contains $0$, $=$, the succession function $\rightarrow$ and may contain symbols of recursive predicates. Formulas are constructed using $\land$, $\lor$, $\forall$, and sequents have the form $\theta \rightarrow \Delta$ where $\theta$, $\Delta$ are lists of formulas.

The degree of a formula is the number of (occurrences of) connectives.

The elementary axioms are defined as follows:

a1. $\land a = 0 \rightarrow$; $\land a = a; \land b \rightarrow a = b; a = b, A(a) \rightarrow A(b)$ [here $A(x)$ is an atomic formula; $a$ and $b$ are terms] are elementary axioms.

a2. Recursive definitions of predicates are elementary axioms. For instance, the elementary axioms for the addition predicate $+(x, y, z)$ are three sequents: $\rightarrow (x, 0, x); + (x, y, z) \rightarrow + (x, y, z); + (x, y, z), + (x, y, u) \rightarrow z = u$.

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a3. Sequents obtained by applying the cut \( \Gamma \vdash A \Delta ; \Sigma A \rightarrow \Pi \Gamma \Sigma \rightarrow \Pi \Delta \) to elementary axioms are elementary axioms.

a4. Refinement \( \Gamma \rightarrow \Delta / \Gamma \Gamma \rightarrow \Delta \) of an elementary axiom is an elementary axiom.

An Induction Axiom. V1. \( \theta(A(\alpha), \forall x A(\alpha(x)) \rightarrow A(\alpha(y)), \Delta \), where \( A(z) \) is any formula and \( \alpha(y) \) is a term in which an induction variable \( y \) appears.

Rules of introduction into the antecedent and the succedent. \( \rightarrow \theta \rightarrow A(y), \Delta \rightarrow \theta \rightarrow \forall x A(x, \Delta) \) (with restriction on variables).

\[
\begin{align*}
\rightarrow & . \theta \rightarrow A, \Delta ; \theta \rightarrow B, \Delta \rightarrow \theta \rightarrow A \& \theta, \Delta ; \theta \rightarrow A, \Delta \rightarrow \theta \rightarrow \forall x A(x, \Delta).
\end{align*}
\]

Chain deduction (CD):

\[
\frac{s_k, \ldots, s_2, s_1, s}{s} \tag{1}
\]

where \( k > 1, s_1, s_2, \ldots, s_k, s \) are sequents; and \( s \) is obtained from \( s_1, \ldots, s_k \) by a proof consisting of applications of cuts. CD is described by the tree (see Fig. 1) at whose vertices the sequents \( s_1, \ldots, s_k \) stand in the order described in (1) and whose downward passages occur according to the cut rule. At each sequent of the thread \( s-s_k \) of the tree (going from \( s \) to \( s_k \)), a subtree starts whose threads end, at the top, with antecedents of the CD. To each such subtree there corresponds a cut formula cut on the thread \( s-s_k \). For instance, the sequent \( \theta \Gamma \rightarrow \Delta \Sigma \) corresponds, in our figure, to the subtree \( s_2, \) the antecedents of the cut are \( \theta \rightarrow A \Delta \) and \( \Delta \rightarrow \Sigma, \) the formula of the cut is \( A. \) To each antecedent of CD corresponds a thread connecting it to the thread \( s-s_k. \)

If \( s_1, \ldots, s_k \vdash s \) by CD and \( s' \) is obtained from \( s \) by cancellation, then we write \( s_1, \ldots, s_k \vdash s'. \) By the degree of CD is meant the maximal of degrees of cuts forming the studied CD.

All rules are considered up to a permutation of formulas in the antecedent and the succedent.

By the order of a proof is meant the number of rule applications.

2. Definition of the Proof Ordinal

2.1. The fundamental ordinal (FO) of a proof \( D. \) FO of an elementary axiom is equal to zero. FO of an induction axiom is \( \omega. \) If \( D \) ends with one of the introduction rules, then \( 1 \) is added to FO of the antecedent (the sum of FO of the antecedents for \( \rightarrow \)). In the case of CD, FO is equal to \( \sum_{i=1}^{\Sigma} \omega^{a_i} \), where \( \Sigma \) is the natural (commutative) sum and \( a_i \) are FO of the antecedents.

2.2. Excess of Proof \( D. \) The excess of an axiom is equal to zero. Suppose that a proof \( D \) ends with an application of a rule \( h. \) Let \( r \) denote the maximal of excesses of the antecedents. If \( h \) is an introduction rule, then the excess of the proof \( D \) is \( r. \) If \( h \) is a CD of degree \( k \) and \( k < r, \) then the excess is \( r-1, \) and if \( k \geq r \), then the excess is \( k. \)

2.3. Let \( \text{acnt} \) (read "a step \( t \)") be defined by the equalities: \( \text{acnt} 0 = 0, \text{acnt} (t + 1) = \omega \text{acnt} t, \omega^0 = 1, \) i.e.,

\[
\text{acnt} = \omega^t \}
\]

The ordinal of proof \( D \) is, by definition, \( \text{acnt}, \) where \( a \) is FO of the proof \( D \) and \( t \) the excess of the proof \( D. \) We will prove that the ordinal of a proof is strictly monotone.