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NORMED RINGS GENERATED BY GENERALIZED CONVOLUTIONS

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Normed rings are considered that are generated by the generalized shift operation that occurs in the study of the analytic properties of Urbanik algebras. Possible applications are illustrated by the example of a counterpart of the classical criterion of positive definiteness and by an inversion formula for generalized characteristic functions of Urbanik.

It is well known that in many problems of probability theory there occur operations with properties similar to those of convolutions, for example, a symmetrical convolution or Kingman's convolution [1] which corresponds to tracing of spherically symmetrical random vectors. A natural extension of these convolutions, introduced by Urbanik [2] with the aid of topological and algebraic axioms, is the generalized convolution of distributions on a half-line for which Bingham [3] obtained counterparts of Khinchin's factorization theorems. A complete description of the class $I_0$ in a semigroup of probability measures with Kingman's convolution has been presented by Ostrovskii [4].

Below we shall consider normed rings that occur in the study of the analytic properties of a generalized convolution. For generalized characteristic functions of Urbanik it is possible to obtain with the aid of such an approach an inversion formula and a counterpart of the positive-definiteness criterion.

1. Notation and Auxiliary Results

In this paper we shall use the following notation: $C$ and $L^r(\eta)$ are Banach spaces of bounded continuous functions on $R_+ = [0, +\infty)$ with a sup norm and of functions with an $L^r$-norm that are absolutely integrable at a power $1 \leq r < +\infty$ with respect to a measure $\eta$ on $R_+$; $L(T)$ is a set of finite continuous functions on $T$; $B$ is a set of probability measures on $R_+$; $D$ is a set of finite charges on $R_+$; $E_\vartheta$ is a distribution concentrated at a point $\vartheta \geq 0$; for $x \in L^r(\eta)$ we denote by $x'\eta$ a charge $X$ such that

$$X(A) = \int_A x(u) \eta(du)$$

for any Borel subset $A$ of $R_+$. For $a \geq 0$ we denote by $H_a$ an operator of multiplicative shift on $D$,

$$(H_aP)(A) = P\left(\frac{A}{a}\right), \quad \text{for} \quad a > 0, \quad (H_0P)(A) = E_\vartheta(A),$$

where $A$ is a Borel subset of $R_+$.

The properties of normed rings, just as those of various other entities related to them, will be specified in the same way as in [5].
According to [2], a commutative and associative operation $\circ$ on $B$ is called a generalized convolution (g.c.) if it is coordinatewise continuous in Levy's metric, homogeneous with respect to $H_a$ for $a \geq 0$;

$$H_a(P \circ Q) = H_aP \circ H_aQ; \quad P, Q \in B;$$

linear, i.e.,

$$(aP + bQ) \circ F = a(P \circ F) + b(Q \circ F)$$

for $a, b \geq 0$, $a + b = 1$, $P, Q, F \in B$; the unit element is denoted by $E_0$:

$$E_0P = P, \quad P \in B;$$

moreover, the following requirement is satisfied: There exists a sequence $e_n \in \mathbb{R}^+$ such that

$$H_{a}^{E_0^{n\alpha}} \rightarrow 0 \neq E_0.$$ (1.1)

Here the degree is interpreted in the sense of g.c.

An algebra $(B, +, \circ)$ is said to be regular if there exists an $h$-continuous nontrivial homomorphism $(B, +, \circ) \rightarrow (\mathbb{R}, +)$. Then the expression

$$\Phi(P, t) = \int_0^\infty \omega(t) P(du); \quad \omega(u) = h(E_0)$$

will be called a generalized characteristic function (g.c.f.) of the measure $P \in B$. The relation $\hat{\Phi}(F, t) = \Phi(P, t)\hat{\Phi}(Q, t)$ will hold if and only if $F = P \circ Q$. The g.c.f. of the distribution $Q$ in (1.1) is $\exp(-ct^\alpha)$, where $c > 0$ and $\alpha > 0$.

2. Generalized Convolution of Charges and Functions

Before going over to the construction of a normed ring, let us define the concept of g.c. of charges and functions. We shall proceed in the same way as in the case of introducing a convolution on locally compact groups (see, for example, [6, Chap. 8]).

For $P \in D$ and $\nu, \omega \in \mathbb{R}^+$ let us write

$$K^\nu P(\omega) = \int_0^\infty \sigma(u, \nu, \omega) P(du), \quad (2.1)$$

where $\sigma(u, \nu, \omega) = (E_u \ast E_\nu)(\omega)$.

**THEOREM 2.1.** For any $P, Q \in B$ we have the equation

$$(P \circ Q)(\omega) = \int_0^\infty K^\nu P(\omega) Q(du). \quad (2.2)$$

**Proof.** For discrete distributions it is possible to verify (2.2) directly. Next, for $P, Q \in B$ let us select sequences $P_n$ and $Q_m$ of discrete distributions such that $P_n \rightarrow P$ and $Q_m \rightarrow Q$. For $f \in C$ we then have

$$\int_0^\infty f(u)(P_n \circ Q_m)(du) = \int_0^\infty T^\nu f(u) P_n(du) Q_m(du), \quad (2.3)$$

where

$$T^\nu f(u) = \int_0^\infty f(\omega) \sigma(u, \nu, d\omega)$$

belongs to $C$ with respect to each variable. By going over in (2.3) to the limit at first with respect to $n$, and then with respect to $m$, we obtain

$$\int_0^\infty f(\omega)(P \circ Q)(d\omega) = \int_0^\infty T^\nu f(\nu) P(du) Q(d\nu),$$

which completes the proof of the theorem.

**COROLLARY 2.1.** For the expression

$$\Phi_1(P) = \int_0^\infty \omega(u) P(du)$$