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GEOMETRY OF NONLINEAR DIFFERENTIAL EQUATIONS

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The paper contains a survey of certain contemporary concepts and results connected with the geometric foundations of the theory of nonlinear partial differential equations. At the base of the account is situated the geometry and analysis on jet spaces, finite and infinite.

INTRODUCTION

1. In this survey we consider certain geometric ideas and results connected with the clarification of the foundations of the theory of nonlinear differential equations. Our approach to the formulation of the problem as well as the method of its presentation is based on the general observation that any "geometry" is the realization of a corresponding "algebra." This kind of "algebra" in the case considered is a system of functors, serving as the theory of differential operators in commutative rings, which it would be appropriate to call simply the differential calculus. For this reason, the paper begins with the necessary digression into the so understood differential calculus, in whose language the rest of the account is conducted. This is not only a question of convenience or style, but is of principal moment, and in particular, in counterweight to Courant's remark*

*Questions connected with partial differential equations of order higher than the first are so varied that the construction of a unified theory does not seem possible. Courant, Partial Differential Equations [Russian translation], Mir, Moscow (1964), p. 159.

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to observe the essential oneness of differential mathematics.

2. The geometric theory of differential equations in the contemporary sense has its origins in the classical works of Lie (cf. [51, 52]), in which there is created a complete theory of first order equations and the foundation is laid for the classical theory of symmetry.

Cartan indicated the importance of invariant methods in the general theory of differential equations and realized his program of invariantization of this theory into the geometric language of vector fields and differential forms. Coordinates, however, still play a noticeable role in Cartan's works, which, for example, is apparent in the fact that he never gave complete proofs of some of his deepest results (cf. [18, 32, 33, 29, 46]).

In the 50's and 60's Cartan's theory underwent further invariantization and refinement. A central place here is occupied by the papers of Kuranishi (cf. [48-50, 45]). Approximately at this time the language of jets introduced by Ehresmann [35] was transformed from the language of "good form" into a useful working language for the theory of differential equations, which, for example, led Goldschmidt to a considerably more satisfactory formulation of the Cartan–Kähler existence theorem and the Cartan–Kuranishi theorem on extensions (cf. [37, 38]). An important moment here was the opening of the mechanism of Spencer cohomology (cf. [30]), which, as is now clear, is an important component part of the differential calculus in the sense indicated above. As a result of this development it became obvious that manifolds of jets are the natural geometric base of the theory of differential equations and this circumstance is one of our starting points. It is useful here to turn our attention to the space of tangent elements, which is the foundation of Lie's theory, and is nothing else than the manifold of jets of the first order. We emphasize, finally, the general impact of the papers of Spencer and Sternberg (e.g., [53]) on the questions we are considering.

3. The relation between the volume of material and the size of the paper defined its style, which is close to telegraphic: formulation of definitions and results plus minimal clarification. For this reason, in many places we could not mention the necessary motivations and examples, indicate the useful applications known at the time the theory developed (propagation of discontinuities, Cauchy's problem, Hamiltonian formalism in field theory) and describe its many and interesting connections with other domains of mathematics and mathematical physics.

Implicitly the paper is divided into three parts. In Secs. 1 and 2 the language is described, Secs. 3-7 are essentially devoted to the geometry of submanifolds of manifolds of jets, and Secs. 8-10 to the geometry of infinitely extended equations. This last "nonclassical" part is the most important, since the objects of the category of nonlinear differential equations turn out to be precisely infinitely extended equations. Here the language of Secs. 1 and 2 is used essentially, while for the second part the ordinary language of the theory of smooth manifolds is completely sufficient.

References to the literature and brief remarks of a priority character are given at the end of each section. Here we do not pretend to completeness and objectivity. In connection with this we refer to the survey [2], where the extensive literature is cited, and also to Forsyth's work [36], from which one can get not only a good representation of the development of the domain of interest to us in the last century, but also derive an inspiring formulation of a whole series of problems. The Bäcklund transformation is by no means the only example of the oblivion of beautiful geometric ideas left us by the classics. Finally, we note the papers [39-42, 31, 47], closely connected with the questions examined below.

4. The geometric foundation of the theory of nonlinear differential equations has been considered during the last several years in the seminar of workers on the mechanico-mathematical faculty of Moscow State University under the guidance of the author. In this paper, essentially, there is elaborated the point of view which results from these discussions, which explains a certain one-sidedness in our references.

Now, when one, apparently, can assume that a return to foundations was justified, the author thanks the participants in this seminar, and in particular, V. V. Lychagin, B. A. Kupershmidt, and I. S. Krasil'shchik for the necessary optimism and enthusiasm.

1. Differential Operators in Commutative Rings

1.1. The hope of constructing differential operations on spaces of sufficiently general character, which arise in the successive constructions of the geometry of differential equations, as well as the general considerations of paragraph 0.2, leads to the necessity of extending the boundaries of the classical differential calculus to smooth manifolds. Such an extension turns out to be possible if one follows the general algebraic point of view described below.