A NONSTATIONARY QUASILINEAR SYSTEM
WITH A SMALL PARAMETER, REGULARIZING A
SYSTEM OF NAVIER–STOKES EQUATIONS

A. P. Oskolkov

We show that the nonstationary quasilinear system

\[ -\varepsilon \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{v} + c_k \frac{\partial \mathbf{v}}{\partial x_k} + \nabla p = f, \quad \text{div} \mathbf{v} = 0, \quad \varepsilon > 0, \quad (A) \]

encountered in the study of aqueous solutions of polymers, regularizes a system of Navier–Stokes equations in the following sense: For each \( \varepsilon > 0 \) the initial-boundary-value problem for system (A) has in the large a unique strong solution (the Ladyzhenskaya solution), while as \( \varepsilon \to 0 \) and for specified smallness conditions on the problem's data, this solution tends to the strong solution of a system of Navier–Stokes equations, whose existence and uniqueness were proved by Ladyzhenskaya.

The existence in the large of a "good" solution of the initial-boundary-value problem for a nonstationary nonlinear system of Navier–Stokes equations has not yet been proved. Ladyzhenskaya [1, 2] and Golovkin [3] have constructed several systems of equations with a small parameter, regularizing a system of Navier–Stokes equations. The initial-boundary-value problems for these regularized systems are solvable in the large in "good" classes of functions, but as the parameter tends to zero the solutions of these regularized systems do not converge, in general, to "good" solutions of the system of Navier–Stokes equations. In the present paper we offer one more system of equations with a small parameter, regularizing a nonstationary system of Navier–Stokes equations. This system, as also the new equations of motion of a viscous liquid, proposed by Ladyzhenskaya and Golovkin, has a definite physical meaning: It describes the laminar motion of aqueous solutions of polymers [4, 5]. The regularizing system of Navier–Stokes equations being proposed by us is linear (i.e., the regularizing terms have the form of a linear differential operator with a small parameter) and, therefore, it has specific advantages in the numerical solution of the problems of the flow of a viscous liquid.

Let \( \Omega \) be a bounded region in a three-dimensional Euclidean space, \( \partial \Omega \) be its boundary; \( QT = \Omega \times [0, T] \), \( 0 < T < \infty \), \( \partial QT = \partial \Omega \times [0, T] \) is the lateral surface of cylinder \( QT \). In \( QT \) we consider the following nonstationary quasilinear system with a small parameter \( \varepsilon > 0 \):

\[ L\mathbf{v} = -\varepsilon \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{v} + c_k \frac{\partial \mathbf{v}}{\partial x_k} + \nabla p = f(x, t), \quad \text{div} \mathbf{v} = 0. \quad (1) \]

For \( \varepsilon = 0 \) system (1) degenerates into a system of Navier–Stokes equations

\[ \frac{\partial \mathbf{v}}{\partial t} - \nabla \cdot \mathbf{v} + c_k \frac{\partial \mathbf{v}}{\partial x_k} + \nabla p = f(x, t), \quad \text{div} \mathbf{v} = 0. \quad (2) \]

For system (1) we shall solve the first initial-boundary-value problem corresponding to it, i.e., in the cylinder \( QT \) we shall seek its solution \( \mathbf{v}, p \), satisfying the following initial-boundary conditions:

\[ \mathbf{v} \big|_{t=0} = a(x), \quad \mathbf{v} \big|_{\partial QT} = 0. \quad (3) \]

A strong solution of the initial-boundary-value problem (1), (3) is the function \( v(x, t) \in L^2(Q_T) \), for which \( v^2, v_x, v_t, v_{xt} \in L^2(Q_T) \) and which satisfies the integral identity
\[
\int_{Q_T} (v \Phi + v_x \Phi_x + \eta x v \Phi_x - \eta x v \Phi_{xx}) dQ = \int_{Q_T} f \Phi dQ, \quad 0 < t \leq T, \tag{4}
\]
for any \( \Phi(x, t) \in \mathring{J}^{(1)}_l(Q_T) \equiv \mathring{J}^{(0)}_l(Q_T) \cap \mathring{J}_l(Q_T). \)

The strong solution of problem (1), (3) is determined uniquely. Indeed, suppose that problem (1), (3) has two strong solutions \( v', v'' \). Then their difference \( \omega(x, t) \equiv v' - v'' \) satisfies the integral identity:
\[
\int_{Q_T} \left[ (\omega \Phi + \omega_x \Phi_x + \eta x \omega \Phi_x) \right] dQ = 0, \quad 0 < t \leq T. \tag{5}
\]

We set \( \Phi = \omega \). Then, allowing for \( \omega |_{t=0} = 0 \), for any \( t \in (0, T] \) we obtain the following equality
\[
\frac{1}{2} \int_{\mathcal{Q}_t} \omega^2(x, t) dx + \frac{\varepsilon}{2} \int_{\mathcal{Q}_t} \omega_t^2(x, t) dx = \int_{\mathcal{Q}_t} (\omega \omega' + \omega \omega'') \omega_t dQ. \tag{6}
\]

Now we make use of the well-known inequality
\[
\|u\|_{L^3(\Omega)} \leq C_0 \|u_x\|_{L^3(\Omega)}, \tag{7}
\]
valid for any function \( u(x) \in W^{1,3}(\Omega) \) and for any bounded region \( \Omega \). Then, by estimating the right-hand side with the aid of the H"older inequality and of inequality (7), we find
\[
\left( \int_{\mathcal{Q}_t} \omega^2 \omega_t dQ \right)^{1/4} \left( \int_{\mathcal{Q}_t} \omega_t^2 dQ \right)^{1/4} \leq C \omega \omega' + \omega \omega'', \tag{8}
\]

We set \( y(t) = \int_{\mathcal{Q}_t} \omega^2 dQ \). Then from (6) and (8) we obtain the following differential inequality:
\[
\frac{\varepsilon}{2} y'(t) \leq Cy(t), \quad 0 < t \leq T, \quad y(0) = 0, \tag{9}
\]
hence follows \( y(t) = 0 \) and, therefore, \( v' = v'' \).

In the present paper it will be shown that system (1) regularizes the system (2) of Navier—Stokes equations in the following sense: 1) if \( f(x, t), \eta f(x, t) \in L^2(Q_T) \), \( \varepsilon(\varepsilon) \in W^{1,3}(Q) \cap \mathring{J}_l^{1/2}(Q) \), then for each \( \varepsilon > 0 \) the initial-boundary-value problem (1), (3) has the strong solution \( v^\varepsilon(x, t) \); 2) under specified smallness conditions on the problem's data, viz., on \( f(x, t), \varepsilon(\varepsilon) \), and on the size of cylinder \( Q_T \), under which the initial-boundary-value problem (2), (3) for the system of Navier—Stokes equations has a strong solution \( v(x, t) \) (a solution in the sense of Ladyzhenskaya), the strong solution \( v^\varepsilon(x, t) \) of problem (1), (3) goes as \( \varepsilon \to 0 \) into the strong solution \( v(x, t) \) of problem (2), (3) for the system of Navier—Stokes equations.

We recall that the strong solution of the initial-boundary-value problem (2), (3) is defined as the function \( v(x, t) \in L^2(Q_T) \) for which \( v^2, v_x, v_t \in L^2(Q_T) \) and which satisfies the integral identity
\[
\int_{Q_T} (v \Phi + v_x \Phi_x - \eta x v \Phi_x) dQ = \int_{Q_T} f \Phi dQ, \quad 0 < t \leq T, \tag{10}
\]
for any \( \Phi(x, t) \in \mathring{J}^{(1)}_l(Q_T) \). As is well known, the strong solution of problem (2), (3) is unique.

The work is based on a priori estimates of the solutions of problem (1), (3). These estimates are derived from the following equalities:
\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2(Q_T)}^2 + \frac{\varepsilon}{2} \frac{d}{dt} \|v_x\|_{L^2(Q_T)}^2 + \|v_t\|_{L^2(Q_T)}^2 = (f, v)_l(Q_T), \quad 0 < t \leq T, \tag{11}
\]
\[
\frac{1}{2} \frac{d}{dt} \|v_t\|_{L^2(Q_T)}^2 + \frac{\varepsilon}{2} \frac{d}{dt} \|v_x\|_{L^2(Q_T)}^2 + \|v_{xt}\|_{L^2(Q_T)}^2 \geq (v_t v_x v_t)_l(Q_T), \quad 0 < t \leq T, \tag{12}
\]
which in their own turn are derived with the aid of an integration by parts from the equalities
\[
\int_Q L_v v v dx = \int_Q f v v dx, \quad 0 < t \leq T. \tag{13}
\]

In obtaining these estimates we shall use inequality (7) and the following statement, sometimes called the Gronwall lemma [1]: If on \( (0, T] \) the function \( y(t) \) satisfies the differential inequality
\[
\frac{\varepsilon}{2} y'(t) \leq Cy(t), \quad 0 < t \leq T, \quad y(0) = 0, \tag{14}
\]
then follows \( y(t) = 0 \) and, therefore, \( \Phi = \omega \) for any function \( u(x) \in W^{1,3}(\Omega) \) and for any bounded region \( \Omega \). Then, by estimating the right-hand side with the aid of the H"older inequality and of inequality (7), we find
\[
\left( \int_{\mathcal{Q}_t} \omega^2 \omega_t dQ \right)^{1/4} \left( \int_{\mathcal{Q}_t} \omega_t^2 dQ \right)^{1/4} \leq C \omega \omega' + \omega \omega'', \tag{15}
\]

We set \( y(t) = \int_{\mathcal{Q}_t} \omega^2 dQ \). Then from (6) and (8) we obtain the following differential inequality:
\[
\frac{\varepsilon}{2} y'(t) \leq Cy(t), \quad 0 < t \leq T, \quad y(0) = 0, \tag{16}
\]

hence follows \( y(t) = 0 \) and, therefore, \( v' = v'' \).