A WEAK-INVERTIBILITY CRITERION IN SPACES OF ANALYTIC FUNCTIONS SEPARABLE BY GROWTH CONSTRAINTS

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1°. Let $X$ be a space of analytic functions defined in the unit disk $D = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. We denote by $\mathcal{Z}$ the identity map of $D$ into itself and assume that $\mathcal{Z}:f \in X \Rightarrow \mathcal{Z}f \in X$. The present article is concerned with the description of functions $f, g \in X$, admitting the polynomial approximation (in the sense of convergence in $X$)

$$q = \lim_{n \to \infty} \rho_n f$$

($\rho_n \in \mathcal{P}$, $\mathcal{P}$ is the set of all polynomials of $\mathcal{Z}$) for any function $q, g \in X$. Functions $f$ having this property are called [1-3] weakly invertible in the space $X$. The investigation of this topic was begun by Keldysh [4] and Beurling [5] and was subsequently continued by many mathematicians (a detailed bibliography may be found in [3]).

Here we consider the spaces $X = A(\lambda)$ consisting of all functions $f$ regular in $D$ such that

$$\lambda(\mathcal{Z}(f)) = \mathcal{Z}(\lambda(f)),$$

where $\lambda$ is a positive increasing weight on the interval $[0, 1)$. Given certain constraints on the weighting function $\lambda$, it is possible to deduce almost-conclusive sufficient conditions for weak invertibility in the space $A(\lambda)$.

We add (see also [3]) that the weak invertibility of an arbitrary function $f, g \in X$ such that $\lambda(\mathcal{Z}(f)) \neq 0$, $|\mathcal{Z}| < 1$, is equivalent to "spectral analyzability" for the operator adjoint to multiplication by $\mathcal{Z}$.

2°. Description of the Results. It has been proved in [3] that (under definite "regular variation" conditions on the weight $\lambda$) for all bounded nonvanishing functions to be weakly invertible in $A(\lambda)$ it is necessary that
In this article we verify the converse statement (Theorem 3 in §4): The divergence of the integral
(0.1) (under definite "regular variation" conditions on \( \lambda \)) induces weak invertibility in \( A(\lambda) \) not only
of all bounded functions \( f, \frac{f(s)}{s} \neq 0, |s| < 1 \), but also of exponential-growth functions \( \int_1^\infty \lambda^{-s} \log \lambda \sim O((1-\epsilon) \lambda) \)
as \( |s| \to 1 \) for some \( \epsilon > 0 \) and even of functions having a "regular" majorant \( \int_1^\infty \lambda^{-s} \in \lambda(1+|s|) \)
with the property \( \int_1^\infty \lambda^{-s} \log \lambda \sim O(1) \) as \( \lambda \to +\infty \). We note at once that the indicated "regularity" constraints on
the weighting functions are tantamount to the requirement of convexity of the first derivative of \( \log \lambda \)
or of \( \log \log \lambda \); the pertinent details are given below in subsection 4.°

An analogous integral condition for weak invertibility is also obtained for the case of rapidly
growing weighting functions. The problem is stated here as follows: The class of functions \( A(\lambda) \) is
given, and a majorant \( \lambda \) is sought for which all functions \( f, \frac{f(s)}{s} \in A(\lambda), \frac{f(s)}{s} \neq 0, |s| < 1 \), are weakly
invertible in the space \( A(\lambda) \). The result can be written in the following form (see Theorem 2 and
the corollary thereto): For all functions \( f, \frac{f(s)}{s} \in A(\lambda), \frac{f(s)}{s} \neq 0, |s| < 1 \) to be weakly invertible in \( A(\lambda) \)
it is sufficient that \( \log \lambda = \log \lambda + h(\log \lambda) \) and

\[
\int_1^\infty \frac{h(u)}{u} \log u^{1/2} du = +\infty,
\]

where \( \infty \) is a convex function for which \( \mu \gamma \in \infty(\mu) \). Here also "regularity" of the growth of the function \( \mu \log \lambda \) is deemed crucial. The behavior of the function \( \infty \) can be qualitatively characterized as
follows: Its growth is weaker, the larger the growth rate of the majorant \( \lambda \), but with \( \infty(\lambda) \searrow \lambda \)
and \( \int_1^\infty \lambda^{-s} \log \lambda < +\infty \). We list some representative cases of growth of the function \( h \) where the weak-invertibility requirement is met:

(a) If \( \mu = \log \lambda \) has [with respect to \( (1-s) \) ] exponential order \( \delta > 0 \), then it is sufficient
to take \( h(\mu) = u^\delta \); \( \delta > 0 \).

(b) If \( \mu \) has infinite order with respect to \( (1-s) \), then it is sufficient to put \( h(\mu) = u^\delta \); \( \delta > 0 \).

(c) If \( \int_1^\infty (\mu(\mu))^\delta du = +\infty \) (roughly: \( \lambda \approx \exp \exp - \frac{1}{1-s} \)), then it is possible to put \( h(\mu) = u^\delta \).

(d) If \( (1-s) \log \mu(\mu) \sim (1-s) \log \log \mu(\mu) > \text{const} > 0, n > 1 \), then it is sufficient to have \( h(\mu) \geq u^n \log u \).

It is important to re-emphasize that all propositions stated in the article are proved only under
the condition of sufficient "regularity" of the majorant \( \lambda \).

It is demonstrated in §5 in examples of special rapidly growing majorants \( \lim_{s \to 0} (1-s) \log \log \lambda(s) = +\infty \) that the exponential order of \( \frac{f(s)}{s} \) characterizing the behavior of the function \( h \) in condition (0.2)
cannot be decreased in any way to preserve the weak-invertibility property of functions from \( A(\lambda) \)
in the space \( A(\lambda) \).