STABILITY OF SOME CHARACTERIZATION PROPERTIES OF THE EXPONENTIAL DISTRIBUTION

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1. Introduction

The characterization properties of the exponential distribution have been studied in many papers. Some of these papers are dealing also with the stability (robustness) of these properties [1-9].

In this paper we shall consider the stability of characterizing the exponential distribution by the absence of an aftereffect in the mean (the "lack-memory" property). Thus, we obtain stability bounds that sharpen some of the results obtained in [1, 4]. We also analyze the stability in a class of distributions with a monotonic intensity.

2. Stability of Characterizing the Exponential Distribution by the Absence of Aftereffect in the Mean

Let $X$ be a nonnegative random variable (r.v.) with a distribution function (d.f.)

$$F(x) = P(X < x)$$

and a reliability function (r.f.) $G(x) = 1 - F(x)$. Azlarov, Dzhamirzaev, and

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Sultanova [1] have obtained the following characterization of the exponential distribution by the absence of an aftereffect in the mean of first and second order (the $k$-th-order case has been considered by Sakhobov and Geshev, see, for example, [8, Theorem 2.3.2]). If we have for a natural $k$ the equation

$$E\{(X-x)^k/X \geq x\} = EX^k < \infty, \tag{1}$$

then $X$ will have an exponential distribution.

For the case $k = 1, 2$, the stability of the result has been studied in [1] as follows.

If the condition

$$\sup_{x \geq 0} |E\{(X-x)^k/X \geq x\} - a| < \varepsilon, \ a > 0, \tag{2}$$

is satisfied, then we shall have $\varepsilon$-proximity between the distribution function $F$ and the exponential distribution in a uniform metric.

It is easy to see that condition (2) is too restrictive, since it implies the existence of an exponential moment of the random variable $X$. Klebanov and Yanushkyavichene [4] have suggested the use of a condition weaker than (2), i.e., absence of a $K$-th-order aftereffect in the mean:

$$\sup_{x \geq 0} |E\{(X-x)^k/X \geq x\} - EX^k| \cdot P(X \geq x) \leq \varepsilon. \tag{3}$$

Similarly to condition (3), let us write for any numbers $k = 1, 2, \ldots$ and $\lambda > 0$ the formula

$$H_k(x; \lambda) = G(x)[E\{(X-x)^k/X \geq x\} - k!/\lambda^k], \ x \geq 0. \tag{4}$$

Then the functional

$$LM_{k, \lambda}(G) = \sup \{H_k(x; \lambda); x \geq 0\}$$

will specify the degree to which the distribution $F$ has no aftereffect. Let us write

$$B(x; \lambda) = G(x) - \exp(-\lambda x), \ \lambda > 0, \ x \geq 0. \tag{5}$$

Then the functional

$$Exp_{\lambda}(G) = \sup \{B(x; \lambda); x \geq 0\}$$

will be a uniform measure of the proximity between the distribution $F$ and an exponential distribution with a parameter $\lambda$.

**Theorem 1.** For any $\lambda > 0$ and a reliability function $G$, we have the inequality

$$Exp_{\lambda}(G) \leq 2LM_{k, \lambda}(G). \tag{6}$$

**Proof.** Let $LM_{k, \lambda}(G) \leq \varepsilon < \infty$. By virtue of definition (4) we have an equation

$$H_1(x; \lambda) = \int_{\lambda}^{\infty} G(u) \, du - \lambda^{-1} G(x). \tag{7}$$

From the inequality $H_1(x; \lambda) \leq \varepsilon$ it follows that the first moment $\mu_1 = EX$ is finite, and that

$$|\mu_1 - \lambda^{-1}| \leq \varepsilon.$$