


EVALUATION OF THE NORMAL DISTRIBUTION FUNCTION

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Known methods are described for evaluating the multivariate normal distribution function in general and in a number of special cases. Methods are presented for evaluation of the one-dimensional normal distribution function and its inverse.

INTRODUCTION

The object of the present survey is an exposition of the basic known methods pertaining to the evaluation of multivariate and one-dimensional normal distribution functions which have been described in the literature up to the middle of 1978. In spite of the fact that a great number of works have been devoted to the investigation of this practically important problem, in the general case a satisfactory solution of it has so far not been found. From a practical point of view the problem of evaluating the normal distribution function of dimension not exceeding three and the problem of computing the probability of the first quadrant have been satisfactorily solved.

The following notation is used in the present survey: \( m = (m_1, \ldots, m_n)' \), mathematical expectation vector; \( \Sigma = (\sigma_{ij}) \), covariance matrix; \( \Sigma^{-1} = (\xi_{ij}) \), matrix inverse to the covariance matrix; \( \varphi_n(x; m, \Sigma) \), density of the n-dimensional normal distribution with mathematical expectation vector \( m \) and covariance matrix \( \Sigma \) (\( \Sigma \) is positive definite);

\[
\Phi_n(x; m, \Sigma) = \int_{y < x} \varphi_n(y; m, \Sigma) \, dy \tag{0.1}
\]

n-dimensional normal distribution. The inequality \( y < x \) means that all elements of the vector \( y \) are less than the corresponding elements of the vector \( x \). The expression on the right side of (0.1) is meaningful for a positive definite matrix \( \Sigma \);

\[
\Phi^*_n(x; m, \Sigma) = \int_{y > x} \varphi_n(y; m, \Sigma) \, dy \tag{0.2}
\]

\( R = (\rho_{ij}) \), correlation matrix, \( \rho_{11} = 1 \); \( \Phi_n(x; 0, R) \), standard n-dimensional normal distribution function; \( \Phi(x) = \Phi_1(x; 0, 1) \), one-dimensional normal distribution function; \( \varphi(x) = \varphi_1(x) \), density function of the one-dimensional normal distribution; \( \varphi(t) = \varphi^{-1}(t) \), a quantile of the normal distribution; \( \Phi(x_1, x_2; \rho_{12}) = \Phi_2((x_1, x_2)'; 0, R) \), two-dimensional normal distribution function; \( K_0(R) = \Phi^*_n(0; 0, R) \), probability of the first quadrant for the standard normal distribution; \( R^* \), correlation matrix with equation correlation coefficients \( \rho_{ij} = \rho \), \( i \neq j \);

\[
P_s(\phi) = K_s(R^*). \]

In the survey, with the exception of certain special cases, we consider a nondegenerate normal distribution.

1. Multivariate Normal Distribution of General Form

We assume with no loss of generality that the dispersions of the components of an \( n \)-dimensional normally distributed vector are equal to 1, i.e., \( \Sigma = R \), and their mathematical expectations are equal to zero. Plackett [106] (see also [18]) proposed a method of transforming the right side of (0.2) to a sum containing integrals of lower multiplicity. This method consists in the following.

We introduce a correlation matrix \( K = (\rho_{ij}) \) and consider the matrices \( K \) and \( R \) as elements of an \( n(n - 1)/2 \)-dimensional space. We introduce the matrix

\[
K_t = (\rho_{ij}(t)) = tR + (1 - t) K,
\]

which is a correlation matrix for \( t \in [0, 1] \). From the formula for the total differential it follows that

\[
\Phi_n^*(x; 0, R) = \Phi_n^*(x; 0, K) + \sum_{i<j} \frac{\partial \Phi_n^*(x; 0, K_{ij})}{\partial \rho_{ij}(t)} d\rho_{ij}(t).
\]

(1.1)

Suppose that the matrix \( R \) is partitioned into blocks,

\[
R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},
\]

where \( R_{11} \) is a square matrix of second order. Similarly, we represent the matrix \( C = R^{-1} \) in the form

\[
C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.
\]

We decompose the vector \( x = (x_1, \ldots, x_n) \) into two parts: \( x = (x^{(1)}, x^{(2)}) \), where \( x^{(1)} = (x_1, x_2)^t \). We have the formula

\[
\frac{\partial \Phi_n^*(x; 0, R)}{\partial \rho_{11}(t)} = \phi(x_1, x_2; \rho_{11}) \Phi_n^{*-2}(x^{(2)}; b, C_{22}^{-1}),
\]

(1.2)

where \( b = (b_3, \ldots, b_n)^t = R_{21}R_{11}^{-1}x^{(1)} \). This formula follows from the following formulas:

\[
\frac{\partial \Phi_n^*(x; 0, R)}{\partial \rho_{11}} = \frac{\partial \Phi_n(x; 0, R)}{\partial x_1},
\]

\[
\frac{\partial \Phi_n(x; 0, R)}{\partial x_1} = \frac{\partial \Phi_n(x; 0, R)}{\partial x_1} \int_{x_2}^{\infty} \cdots \int_{x_n}^{\infty} \phi_n((x_1, x_2, y_3, \ldots, y_n); 0, R) dy_3 \cdots dy_n,
\]

(1.3)

The correlation matrix \( K \) on the right side of Eq. (1.1) can be chosen, for example, such that it differs from the matrix \( R \) only by the element \( \tau_{n-1,n} \) and has rank \( n - 1 \). In this case the sum on the right side of (1.1) contains only one term, and all the integrals have order \( n - 1 \) [taking account of the formula (1.2)]. Such a value \( \tau_{n-1,n} \) can be found as one of the solutions of the quadratic equation

\[
K_{21}R_{11}^{-1}K_{12} = 1.
\]

By applying the procedure repeatedly, it is possible to reduce the order of the integrals by approximately twice. Actually, this method can be applied, apparently, to evaluate the normal distribution function up to fifth order.

2. In the work of Kendall [83] the two-dimensional normal distribution function is expanded in a series of tetrachoric functions

\[
\tau_r(x) = \frac{H_r(x)}{\sqrt{r!}},
\]

where \( H_r(x) \) is the \( r \)-th Hermite polynomial. Using the results of this work, Dutt [59] obtained an expansion of general form

\[
\Phi_n(x; 0, R) = \sum_{i=0}^{n-1} \sum_{i=0}^{n} A_{i_1 \ldots i_{n-1,n}}
\]

(1.4)