LIMIT THEOREMS FOR NONNEGATIVE-DEFINITE QUADRATIC FORMS
IN CERTAIN DEPENDENT RANDOM VARIABLES

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Let \( \sum_{i=1}^{n} a_{ij}^\nu_1 \xi_{ij} \), \( n = 1, 2, \ldots \), be nonnegative-definite quadratic forms, \( a_{ij}^\nu \) be non-random numbers, \( a_{ij}^\nu = a_{ji}^\nu \), \( \mu_i = 1, n \), \( n = 1, 2, \ldots \), be random variables, and \( M(\xi_{i1}/\sigma_i^\nu) = 0 \), where \( \sigma_i^\nu \) is the minimal \( \sigma \)-algebra with respect to which the random variables \( \xi_{i1}, \ldots, \xi_{in} \) are measurable.

In this article we find conditions under which the distribution of the quadratic form \( \sum_{i=1}^{n} a_{ij}^\nu \xi_{ij} \) can be approximately replaced for large \( n \) by the distribution of the quadratic form \( \sum_{i=1}^{n} a_{ij}^\nu \xi_{ij} \), where \( \gamma_{in}, i = 1, n \) are certain independent random variables. We note that limit theorems for some quadratic forms made up of stationary sequences were looked at in [1]. Limit theorems for sums \( \sum_{i=1}^{n} \xi_{in} \) were studied in many papers (see, e.g., [2-5] and other references).

We denote by \( \eta_{in}, i = 1, n \) independent random variables distributed according to the normal law with zero mean vector and covariance matrix \( 2A_n, A_n = (a_{ij}), i, j = 1, n \). The random variables \( \eta_{in} \) do not depend on the variables \( \xi_{in}, i = 1, n \), \( n = 1, 2, \ldots \).

THEOREM 1. Let

\[
\lim_{n \to \infty} \lim_{k \to \infty} M \sum_{i=1}^{n} \eta_{in} M(\xi_{in}/\sigma_{in}) \exp \left\{ -\alpha \sum_{i=1}^{n} M(\xi_{in}/\sigma_{in}) \eta_{in}^2 \right\} = 0, \tag{1}
\]

where \( \sigma_{kn} \) is the minimal \( \sigma \)-algebra with respect to which the variables \( \xi_{i1}, \ldots, \xi_{in}, \eta_{in}, i = 1, n \) are measurable. For each \( s \in [0, S] \), for an arbitrary positive number \( S > 0 \), and \( \alpha > 0 \),

\[
\lim_{n \to \infty} M \sum_{i=1}^{n} \left| M(\beta_{in}/\sigma_{kn}) \right|^2 = 0, \tag{2}
\]

where \( \beta_{in} = (e^{is\gamma_{kn}} - is\gamma_{kn} - 1), \nu_{kn} = \mathbb{E}_{in} \eta_{in} \exp \left( -\alpha \sum_{i=1}^{n} M(\xi_{in}/\sigma_{in}) \right), k = 1, n \),

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left( M(\beta_{in}/\sigma_{kn}) - M(\beta_{kn}/\sigma_{kn}) \right) = 0. \tag{3}
\]

Then, for each \( q \in [0, S^2] \),

\[
\lim_{n \to \infty} \left[ M e^{-q \sum_{i=1}^{n} a_{ij}^\nu \xi_{ij} \eta_{in}} - M e^{-q \sum_{i=1}^{n} a_{ij}^\nu \gamma_{in} \eta_{in}} \right] = 0, \tag{4}
\]

where \( \gamma_{in}, i = 1, n \) are independent random variables distributed according to an infinitely divisible law with characteristic function \( M \exp \left( is\gamma_{kn} \right) = \exp \left[ M \exp \left( is\gamma_{kn} \right) - 1 \right] \).

Proof. Clearly \( M \exp \left( -q \sum_{i=1}^{n} a_{ij}^\nu \xi_{ij} \eta_{in} \right) = \mathbb{E}_{in} \exp \left( q \sum_{i=1}^{n} \xi_{in} \eta_{in} \right) \), where \( s = \sqrt{q}, q > 0 \). We
assume that $0 \leq s \leq S$. Making use of condition (1), we obtain that
\[
\lim \lim_{a \to 0, n \to \infty} \left| \exp \left( is \sum_{k=1}^{n} \eta_{k n} \pi_{k n} \right) - f_{n}(s) \right| = 0,
\]
where $f_{n}(s) = M \exp \left( is \sum_{k=1}^{n} \rho_{k n} \right)$.

We consider the expression
\[
g_{n}(s) = M \left[ \exp \left( is \sum_{k=1}^{n} \eta_{k n} - \sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \right) - 1 \right] m_{n}(s).
\]
Here $m_{n}(s) = \exp \left( \sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \right)$. It is clear that
\[
\sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \leq \frac{s^{2}}{2} \sum_{k=1}^{n} M \left( \eta_{k n}^{2} / \sigma_{k n} \right) \exp \left\{ - 2a \sum_{l=1}^{n} M \left( \eta_{l n}^{2} / \sigma_{l n} \right) \right\}.
\]
(6)

Since, for every sequence of real numbers $a_{i} \geq 0$, $i = 1, n$,
\[
a_{1}e^{-a_{1}} + a_{2}e^{-a_{2} - a_{1}} + \ldots + a_{n}e^{-a_{n} - \ldots - a_{1}} \leq \sum_{k=1}^{n} \left[ e^{-a_{1} - \ldots - a_{k}} - e^{-a_{k} - \ldots - a_{n}} \right] = e^{-a_{1}} - e^{-a_{2} - \ldots - a_{n}},
\]
then, using (6), we have that
\[
\sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \leq s^{2} / 2a.
\]
(7)

Therefore, we get, using (3), that
\[
g_{n}(s) = M \left[ \exp \left( is \sum_{k=1}^{n} \eta_{k n} - \sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \right) - 1 \right] \varphi_{n}(s) + o(1),
\]
where $\varphi_{n}(s) = \exp \left( \sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \right)$.

We express (8) in the following form:
\[
g_{n}(s) = M \sum_{l=1}^{n} \left[ \exp \left( is \sum_{k=1}^{n} \eta_{k n} - \sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \right) - 1 \right] \theta_{l} \varphi_{n}(s) + o(1),
\]
where $\theta_{l} = \exp \left( is \sum_{k=1}^{n} \eta_{k n} - \sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \right)$.

In this and in other analogous formulas, we consider that $\theta_{n} \equiv 0$. Clearly, (9) equals
\[
g_{n}(s) = \sum_{l=1}^{n} M \left[ 1 + M \left( \beta_{l n} / \sigma_{l n} \right) \right] e^{-M \beta_{l n} / \sigma_{l n}} \theta_{l} \varphi_{n}(s) + o(1).
\]
Hence, using (2), (7), and (3), we get $\lim_{n \to \infty} \left| \exp \left( is \sum_{k=1}^{n} \eta_{k n} - \sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \right) - 1 \right| = 0$. But then, from (5) in view of (1), it follows that $\lim_{n \to \infty, a \to 0} \left| M \exp \left( is \sum_{k=1}^{n} \eta_{k n} - \sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \right) - M \exp \left( is \sum_{k=1}^{n} \eta_{k n} \right) \right| = 0$.

Hence also from (5) follows (4). Theorem 1 is proved.

We note that $\sum_{i,j=1}^{n} \xi_{i j} \theta_{i j} M_{n} n i$ are quadratic forms of independent random variables, and to find their limit distributions we can use the results of [6-7].

**THEOREM 2.** Let condition (1) hold, and let
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \eta_{k n}^{2} \left( M \sigma_{k n}^{2} / \sigma_{k n} \right) - M_{\sigma_{k n}}^{2} = 0, \quad \sum_{k=1}^{n} M_{\sigma_{k n}}^{2} \eta_{k n}^{2} < \infty
\]
(10)

Also let the Lindeberg condition be satisfied: for each $\tau > 0$,
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \int_{|x| > \tau} x^{2} dP \left( \sqrt{H_{k n}} \pi_{k n} < x \right) = 0.
\]
(11)

Then
\[
\lim_{n \to \infty} \left[ M e^{-\eta_{1 n}^{2} / 2} \left( \sqrt{H_{1 n}} \pi_{1 n} \right)^{2} - \det \left( I + 2q \left( \sigma_{i j} \sigma_{j i} \right)^{1/2} \right) \right] = 0,
\]
(12)

where $q \geq 0$, $\sigma_{i} = \sqrt{\mathbb{E} \left( \pi_{i}^{2} \right)}$.

**Proof.** It is clear that
\[
\sum_{k=1}^{n} M \left( \beta_{k n} / \sigma_{k n} \right) \leq \sum_{k=1}^{n} \int_{|x| > \tau} x^{2} dP \left( \sqrt{H_{k n}} \pi_{k n} < x / \sigma_{k n} \right) + \tau \left| \frac{\eta_{k n}}{\sigma_{k n}} \right| \sum_{k=1}^{n} \eta_{k} M \left( \sigma_{k n}^{2} / \sigma_{k n} \right).
\]
In this and in the following formulas, we consider that $\eta_{k} / \sqrt{\sigma_{k n}} = 0$ if $\sigma_{k k} = 0$. Hence, making use of conditions (10) and (11), we obtain that expression (2) is valid.