SOME PROPERTIES OF FUNCTIONS OF CLASS S

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1. Introduction. By the letters N, R, and C we shall denote the sets of all natural, real, and complex numbers, respectively; let A be the empty set, and \( \Delta_{\xi} \) an open circle \( |\xi| < r, \xi \in C, r > 0 \). We shall consider a linear complex-valued functional \( a_{h}(f) = \frac{1}{k!} \left( \frac{d^{k}}{dz^{k}} f(z) \right)_{z=0} \forall k \in N \) on the set of all functions regular at the point \( z = 0 \). The class S consists of the set of all functions that are univalent and regular in \( \Delta_{\xi} \) and are normalized by the expansion

\[
f(z) = z + a_{1}(f)z^{2} + ... + a_{n}(f)z^{n} + ...
\]  

(1)

The class K consists of the set of all \( K_{\theta} \)e functions \( f_{\theta}(z) = z(1 + e^{i\theta}z)^{-2}, \theta \in R \).

It is evident that \( K \subset S \). Let us introduce the notation \( a_{\infty} = \sup_{n \in N} |a_{n}(f)| \forall n \in N \). It is well known [1-4] that \( a_{n} < \infty \forall n \in N \). The compactness in itself of the class S ensures the existence of functions \( f_{0} \in S \) such that

\[
|a_{n}(f_{0})| = a_{n} \forall n \in N.
\]  

(2)

For any \( n \in N \) we shall define the class \( S(n) \) by the set of all functions \( f \in S \) whose \( n \)-th coefficient satisfies (2).

Let us introduce the notation

\[
S(n, m) = S(n) \cap S(m), \quad 2 \leq n < m, \quad n \in N, \quad m \in N.
\]

It is well known [5-7] that any function \( f(z) \in S(n) \) satisfies a differential functional equation of the form

\[
P_{n}(w) \left( \frac{dw}{dz} \right)^{2} = Q_{n}(z) \left( \frac{dz}{z} \right)^{2}, \quad P_{n}(w) = \sum_{v=2}^{n} \frac{a_{v}}{a_{n}} \frac{1}{w^{v-1}},
\]

\[
Q_{n}(z) = \sum_{v=1}^{n-1} \left( \frac{va_{v}}{a_{n}} \right) z^{v-n} + (n - 1) z^{n} + \sum_{v=1}^{n-1} \left( \frac{va_{v}}{a_{n}} \right) z^{n-v}, \quad a_{v} = a_{v}(f),
\]

\[
a_{v}^{2} = a_{n}(|f(\xi)|) w = f(\xi).
\]

An equation of the form (3) constructed from the coefficients of a function \( f \in S \) such that \( a_n(f) \neq 0 \) will be called a \( D_n \)-equation associated with the function \( f \). For any \( m, n \in \mathbb{N} \), we shall define the class \( L(m) \) by the set of all functions \( f \in S \) such that \( a_m(f) = 0 \) and \( f \) satisfies the associated \( D_m \)-equation in \( \Delta _{z}^\frac{1}{2} \).

Let us consider the class \( SL(n, m) = S(n) \cap L(m) \), where \( m \neq n, m \in \mathbb{N}, n \in \mathbb{N}, m \geq 2, n \geq 2 \). A function \( w = w(z) \) is algebraic if it satisfies the equation \( P(z, w) = P_{0}(z)w^{n} + \ldots + P_{n}(z) = 0 \), where the \( P_{0}, \ldots, P_{n} \) are polynomials in \( z \), \( P_{0}(z) \neq 0 \) (see [8, pp. 530–543]). The number of branches of the algebraic function \( w = w(z) \) will be denoted by \( q = q(w) \).

II. Formulation of Principal Results. In our notation the well-known hypothesis of Bieberbach [1–4] about the coefficients of functions of class \( S \) takes the form \( S(n) = K, n \geq 2, n \in \mathbb{N} \).

It is then natural to assume that \( S(n, m) = K, 2 \leq n < m \). The following result is of interest from this point of view.

THEOREM 1. Let \( (n, m), 2 \leq n < m, \) be a pair of natural numbers. Then we can have only one of the following two assertions: 
1) \( S(n, m) = K \); 2) \( S(n, m) = \Lambda \).

The author has formulated Theorem 1 in the form of an assertion in [9], where the theorem has been proved for the case \( m = n + 1 \). The complete result has been obtained in [10]. In this paper we shall show that the assertion of Theorem 1 holds under weaker assumptions. In general it is evident that
\[
S(n, m) \subset SL(n, m) \tag{4}
\]
for any \( n \in \mathbb{N}, m \in \mathbb{N}, 2 \leq n < m \). However, we obtain the following result.

THEOREM 2. For any pair \( (n, m), n \geq 2, m \geq 2, m \neq n, m, n \in \mathbb{N}, \), only one of the following two possibilities can be realized:

1. \( SL(n, m) = K \); 2. \( SL(n, m) = \Lambda \).

It is easy to see that Theorem 2 yields the assertion of Theorem 1. Indeed, let the assertion of Theorem 2 be valid. For any pair \( (n, m), 2 \leq n < m \) we then have either \( S(n, m) = \Lambda \) or \( S(n, m) \neq \Lambda \). In the first case there is nothing to prove. In the second case formula (4) yields \( SL(n, m) \supset S(n, m) \neq \Lambda \); hence \( SL(n, m) = K \). Thus \( S(n, m) \subset K \), and since \( e^{i\theta}f(e^{-\theta}z) \in S(n, m) \) together with \( f(z), \forall \theta \in \mathbb{R}, \) it follows that \( S(n, m) = K \).

III. Proof of Theorem 2. At first we shall prove some auxiliary assertions.

LEMMA 1 (see [11, p. 260]). Let \( (n, m) \) be a pair of natural numbers \( n \geq 2, m \geq 2, m \neq n, m, n \in \mathbb{N}, \) such that \( SL(n, m) \neq \Lambda \). Then we have the following assertions:

a) For any function \( f(z) \in SL(n, m) \) there exists an algebraic function \( F_{f}(z) \) such that the restriction of one of its branches to \( \Delta _{z}^\frac{1}{2} \) coincides with \( f(z) \);

b) each branch \( f_{l} \) of the algebraic function \( F_{f} \) is regular at the point \( z = 0 \) and it has in some neighborhood the form \( f_{l}(z) = \mu_{l}^{1}z + \ldots, |\mu_{l}^{1}| = 1, l = 1, 2, \ldots, q(F_{f}) \);

c) if \( f_{1}(z) \neq f_{2}(z) \), then \( \mu_{1}^{1} \neq \mu_{2}^{1}, 1 \leq 1 < j \leq q(F_{f}) \), with \( q(F_{f}) \leq |m - n| \);

d) each branch \( f_{l} \) of the algebraic function \( F_{f}(z) \) is regular at the point \( z = \infty \), and it has in a neighborhood of this point an expansion \( f_{l}(z) = e_{l}^{1}z + \ldots, |e_{l}^{1}| = 1, l = 1, 2, \ldots, q(F_{f}) \);

e) for any \( z \) such that \( 0 < |z| < \infty \) we have \( F_{f}(z) \neq 0 \).

The validity of assertions a, b, d, and e of Lemma 1 follows directly from Babenko's result [11, p. 260]. Therefore we must prove only assertion c.

Proof (Assertion c). Any function \( f(z) \in SL \) satisfies a system of \( D_{n} \) and \( D_{m} \) equations associated with \( f \),
\[
\left( \frac{z}{w} \frac{d}{dz} \right)^{\frac{1}{2}} P_{n}(w) = Q_{n}(z), \quad \left( \frac{z}{w} \frac{d}{dz} \right)^{\frac{1}{2}} P_{m}(w) = Q_{m}(z). \tag{5}
\]
It is easy to see that the mapping \( w = f(z) \in SL \) transforms the zeros of \( Q_{n}(z) \) and \( Q_{m}(z) \) into the zeros of \( P_{n}(w) \) and \( P_{m}(w) \), respectively. Therefore, we can select a region \( \delta_{n}, \delta_{n} \subset \Delta _{z}^\frac{1}{2} \) such that the functions \( P_{n}(w), P_{m}(w), Q_{n}(z), Q_{m}(z) \) do not vanish in this region. Then we have in \( \delta_{n} \) the relation
\[
\frac{P_{n}(w)}{P_{m}(w)} = \frac{Q_{n}(z)}{Q_{m}(z)} \neq \text{const}. \tag{6}
\]