We consider a system of ordinary differential equations containing slow (position) and fast angular variables of the form

$$\frac{dx}{dt} = \varepsilon \alpha(x, \varphi), \quad \frac{dq}{dt} = \omega(x),$$

where \( x \) and \( \varphi \) are \( n \)- and \( m \)-dimensional vectors, respectively, \( \varepsilon \) is a small positive parameter, and the real vector-functions \( \alpha(x, \varphi) \) and \( \omega(x) \) are defined in the region

$$G_{n+m} = \{ x \in D \subset R^n, \; \varphi \in R^m \}$$

and are \( 2\pi \)-periodic with respect to \( \varphi \) there. Systems of this form have been treated by many authors [1-6]. In the present note we give a fundamental method for averaging with respect to fast variables, taking into account relations among frequencies [5], which allows long-periodic harmonics as well as the old terms to remain in the averaged equations.

Let the natural number \( N \) be given and consider the set \( Q \) consisting of \( m \)-dimensional integral vectors \( k \) for which \( |k| = \sum_{i=1}^{m} |k_i| \leq N \). Suppose that \( \bar{Q} \subset Q \). Then we associate with system (1) the system

$$\frac{dx}{dt} = \varepsilon \sum_{i \in \bar{Q}} a_k(\bar{x}) e^{i(k, \varphi)}, \quad \frac{dq}{dt} = \omega(\bar{x})$$

which we call its averaged system. We study the proximity of the slow solutions of (1) and (3) on the time interval \([0, \varepsilon^{-1}]\) under the conditions that \( x(0) = \bar{x}(0), \; \varphi(0) = \bar{\varphi}(0) \).

We assume that \( Q = \bigcup_{i=1}^{4} Q_i \), where

$$Q_1 = \{ k : |(k, \omega(x))| \leq a_1 |k|^{-l} \; \forall x \in D \},$$

$$Q_2 = \left\{ k : \left\| a_k(\bar{x}) \right\| dt \leq a_2 \varepsilon^{-l} \forall t \in [0, \varepsilon^{-1}], \; l_2 < 1 \right\},$$

$$Q_3 = \{ k : \frac{\delta(k, \omega)}{\| \partial(x, \omega) \|} \leq a_3 |k|^{-l} \forall x \in D \},$$

$$Q_4 = \{ k : \left| \frac{\delta(k, \omega)}{|k|^{-l}}, \; \delta(x, \varphi) \right| \geq a_4 \varepsilon^{l}, \; |(k, \omega(x))| \leq a_5 \varepsilon^{l}, \; \varphi \in R^m \}.$$
THEOREM. Suppose that system (1) is such that

1) the vector-functions $a(x, \varphi)$ and $\omega(x)$ satisfy the conditions

$$
\sum_{i=1}^{n} \sup_{x} |a_i(x)| \leq \sigma_0, \max_{\varphi_0, \varphi_1} \sup_{x} \left| \frac{\partial^{|\varphi_0|} a_0 (x, \varphi)}{\partial \varphi_1^{\varphi_1}} \right| \leq \sigma_0^2,
$$

where $i = 1, n, \rho = (\rho_1, \ldots, \rho_m)$, $s = (s_1, \ldots, s_n)$ are integer vectors with nonnegative components, and $a(i)$ is the $i$-th component of $a(x, \varphi)$; 

2) for $N = E\{e^{-C_1}\}$, $Q = \bigcup_{i=1}^{4} Q_i$, where the $Q_i$ are determined by conditions (4)-(7); 

3) $Q_3 \subset \bigcup_{i} \bigcap_{j} Q_i$ and for each vector $k \in Q \cap Q_4$, a condition analogous to (7) is fulfilled, in which the function $\delta(x, \varphi)$ is replaced by

$$
\tilde{\delta}(x, \varphi) = \sum_{k \in Q} a_k(x) e^{i_{k}(\varphi, \varphi_i)} + \sum_{k \in Q} a_k(x) e^{i_{k}(\varphi, \varphi_i)}
$$

4) $l_1 < \frac{1}{2} \cdot l_t < \frac{1}{2}$, $l_t > m + \max(1 + 2l_1, 2 + l_1, l_3)$; 

5) the solution $\tilde{x}(t)$ of the averaged system (3) is contained in the region $D$ together with its $\mu$-neighborhood.

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and $t \in [0, \varepsilon^{-1}]$ we have that

$$
\| x(t) - \tilde{x}(t) \| \leq \sigma_{10} \exp \left\{ \frac{1}{2} t + \frac{1}{2} \right\},
$$

where $\sigma_{10}$ is some constant.

Proof. We put $z(t) = x(t) - \tilde{x}(t)$ in (1); then

$$
\frac{dz}{dt} = \varepsilon \sum_{k \in Q} a_k(x) e^{i_{k}(\varphi, \varphi_i)} + \sum_{k \in Q} a_k(x) e^{i_{k}(\varphi, \varphi_i)} - \sum_{k \in Q} a_k(x) + \varepsilon \sum_{k \in Q} a_k(x) e^{i_{k}(\varphi, \varphi_i)},
$$

where $a_N = \sum_{k \in Q} a_k(x) e^{i_{k}(\varphi, \varphi_i)}$, $R_N = \sum_{k \in Q} a_k(x) e^{i_{k}(\varphi, \varphi_i)}$, $\frac{dz}{dt}$ is a certain average value of $\frac{a_N(x)}{dx}$ in $G_{n+m}$. Taking into account the facts that $k \in Q_3$ and

$$
\left| (k, \varphi) - (k, \varphi) \right| = \left| \int_{0}^{t} \left( (k, \omega(x)) - (k, \omega(x)) \right) d\tau \right| \leq \sigma_0 e^{-k l t + \frac{1}{2} t},
$$

we get that

$$
\| z(t) \| \leq \varepsilon \sum_{k \in Q} \| a_k \| k l t + \frac{1}{2} t \| z(t) \| d\tau + \varepsilon \sum_{k \in Q} \| \frac{a_N}{dx} \| \| z(t) \| d\tau + \varepsilon \sum_{k \in Q} \| a_k \| e^{i_{k}(\varphi, \varphi_i)} d\tau + \varepsilon \sum_{k \in Q} \| a_k \| e^{i_{k}(\varphi, \varphi_i)} d\tau + \varepsilon \sum_{k \in Q} \| R_N d\tau \|
$$

It is well known [6] that if (8) is satisfied, then there exist constants $\sigma_{11}$ and $\sigma_{12}$ such that $\| R_N \| \leq \sigma_{11} N^{-l_t-m}$

$$
\sigma_{11} N^{-l_t-m} + \varepsilon \sup_{k \in Q} \| a_k \| \leq \sigma_{12} \text{ for } l_t > m + l_3.
$$

Since $|dz(t)| \leq \| z(t) \| d\tau$ for $t \in [0, \varepsilon^{-1}]$, it follows by the Gronwall-Bellman lemma that

$$
\| z(t) \| \leq \left[ \sigma_{11} N^{-l_t-m} + \varepsilon \sup_{k \in Q, \omega^{-1}} \left( \left\| \int_{0}^{t} \sum_{k \in Q} a_k(x) e^{i_{k}(\varphi, \varphi_i)} d\tau \right\| + \left\| \int_{0}^{t} \sum_{k \in Q} a_k(x) e^{i_{k}(\varphi, \varphi_i)} d\tau \right\| \right) \right] \exp \left( \sigma_{12} + n \sum_{i=1}^{n} \sigma_{0}^{i} \right).
$$

(11)