FINITE IRREDUCIBLE GROUPS, GENERATED BY REFLECTIONS, ARE MONODROMY GROUPS OF SUITABLE SINGULARITIES

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1. Introduction

1. Groups, Generated by Reflections, and Singularities. The theory of singularities of smooth functions is closely connected with the theory of finite groups, generated by reflections. This connection appears in the following three assertions.

(1) The variety of orbits of the complexification of the action of a finite reflection group is biholomorphically equivalent with the base of a miniversal deformation of the corresponding singularity. Under this isomorphism the variety of nonregular orbits is mapped onto the bifurcation diagram.

(2) A reflection group is isomorphic with the monodromy group of the corresponding singularity.

(3) The isomorphism cited in the first assertion is defined by the period map, i.e., by integration of a holomorphic form defined on the total space of the bundle of hypersurfaces of level zero over the complement of the bifurcation diagram, with respect to a basis in the homology space of the fibre which depends continuously on a point of the base of the bundle.

Finite irreducible groups generated by reflections are classified and exhausted by the following list: $A_n$, $B_n(=C_n)$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$, $I_2(p)$, $H_3$, $H_4$. The majority of them are groups of symmetries of regular polyhedra: $A_n$ of a $n$-dimensional simplex, $B_n$ of a $n$-dimensional cube, $G_2$ of a hexagon, $I_2(p)$, $p > 6$, $p = 5$ of a $p$-gon, $H_3$ of an icosahedron, $F_4$, $H_4$ of the corresponding four-dimensional polyhedra [1, 2].

The singularities corresponding to the indicated groups are denoted by the same letters and are found in [3, 4, 5, 6]. In [4, 7, 8, 9], (1)-(3) are proved for the singularities $A_n$, $B_n$, $E_6$, $E_7$, $E_8$ of functions of an odd number of variables. In [5], (1) and (2) are proved for singularities $B_n$, $C_n$, $F_4$ of functions of an odd number of variables on a manifold with boundary. In [6], (1) is proved for the singularities $G_2$, $I_2(p)$, $H_3$ of functions on a manifold with singular boundary. Recently, O. P. Shcherbak produced a singularity which he called $H_4$, and proved (1) for it.

In this paper (3) is proved for the singularities $B_n$, $C_n$, $F_4$ of functions of an odd number of variables; the singularities cited, corresponding to the groups $G_2$, $I_2(p)$, $H_3$, are different from those cited in [6], but are closely connected with them; for these singularities (1)-(3) are proved. Analogs of (2) and (3) are unknown for the group $H_4$.

2. Symmetric Singularities. In [5] the following interpretation of singularities of functions on a manifold with boundary was used. After passage to the two-sheeted covering, functions on a manifold with boundary become functions which are symmetric with respect to the action of the cyclic group $Z_p$, which changes the sign of one of the coordinates. In the present paper this analogy is extended. We consider singularities of functions, which are symmetric with respect to the cyclic group $Z_p$, and their symmetric miniversal deformations. In this situation $Z_p$ acts on the homology of nonsingular level hypersurfaces of the functions. This action commutes with the natural action of the fundamental group of the complement of the bifurcation diagram on the parameter space of the deformation. Thus, the homology splits into the direct sum of subspaces, which are invariant both with respect to the action of the group $Z_p$, and with respect to the action of the fundamental group.

The groups $G_2$, $I_2(p)$, $H_3$ arise as images of the action of the fundamental group on a suitable invariant subspace in the homology of a suitable symmetric singularity. The period

map is constructed as follows. On all level hypersurfaces of functions there is singled out uniquely a holomorphic form of highest degree. There is singled out a basis which is covariant constant in the Gauss–Manin connection, of a suitable invariant subspace of the homology. The period map relates a point of the complement of the bifurcation diagram to the vector of integrals of the form over the homology classes of the basis, defined up to the action of the monodromy group. One proves that this map extends holomorphically to a map of the parameter space of the deformation into the space of orbits of the complexification of the action of the corresponding group, generated by reflections, and has the properties cited in (1).

In this paper the concept of an equivalent vanishing vector in the homology of non-singular level hypersurfaces of functions constituting a symmetric miniversal deformation is defined, generalizing the concept of a vanishing vector [10]. Equivariant vanishing vectors (with suitable degree of equivariance) for the singularities $A_\mu$, $D_\mu$, $E_\nu$, $E_\phi$, $F_\phi$, $G_\phi$ form a system of roots of the synonomous types; those for the singularities $B_\mu$, $C_\mu$ do the same but for types $G_\mu$, $B_\mu$ respectively. The collection of equivariant vanishing vectors for singularities $I_2(\pi)$, $H_3$ have properties analogous to the properties of systems of roots. Cf. Sec. 2.7 for more details.

The symmetric singularities corresponding to the groups $G_2$, $I_2(\pi)$, $H_3$ arise in the following way from the singularities $G_2$, $I_2(\pi)$, $H_3$ cited in [6] of functions on a manifold with singular boundary. In each case the pair manifold-boundary is isomorphic with the pair consisting of the space of orbits and the space of nonregular orbits of the group of symmetries of a suitable regular polyhedron. After passage to a suitable covering, the functions on the manifold with boundary turn into our functions, which are symmetric with respect to the group of symmetries of the polyhedron, in particular, with respect to the cyclic group of rotations of it.

Cf. [11, 12] also on symmetric singularities.

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2. Formulation of Results

1. Equivalent Monodromies, Vanishing Vectors, and Period Map. Let $G$ be a finite group acting linearly on $\mathbb{C}^n$, $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function at an isolated critical point, symmetric with respect to $G$. A deformation $F: (\mathbb{C}^n \times \mathbb{C}^l, 0 \times 0) \to (\mathbb{C}, 0)$ is said to be a $G$-deformation, if for any $\lambda \in \mathbb{C}^l$ the function $F(\cdot, \lambda)$ is $G$-invariant. A $G$-deformation of the germ $f$ is said to be $G$-equivalent with a deformation induced from $F$ (cf. [3, 13]) for more precision). A versal $G$-deformation with smallest number of parameters is called miniversal. As a miniversal $G$-deformation one can take $F(x, \lambda) = f(x) + \sum \lambda_j \varphi_j(x)$, where $\{\varphi_j\}$ generate a basis in $C[[x_1, \ldots, x_n]]/(\partial f/\partial x) G$, and by the index $G$ we denote $G$-invariant series [13]. We always take $\varphi_1 \equiv 1$.

We choose a sufficiently small ball $B = \{x \in \mathbb{C}^n \mid ||x|| < \varepsilon\}$. Depending on $\varepsilon$ we choose a sufficiently small ball $\Lambda = \{\lambda \in \mathbb{C}^l \mid ||\lambda|| < \delta\}$. We denote by $V_{\lambda}$ the intersection of the zero level hypersurface of the function $F(\cdot, \lambda)$ with the ball $B$. By the bifurcation diagram of the deformation $F$ is meant the subset $\Sigma \subseteq \Lambda$, consisting of those parameters $\lambda$, for which the hypersurface $V_{\lambda}$ is singular. Over $\Lambda \setminus \Sigma$ the manifolds $V_{\lambda}$ form a locally trivial bundle. With this bundle there are associated the cohomology bundle $H^{n-1} \to \Lambda \setminus \Sigma$ with fibre $H^{n-1}(V_{\lambda}, \mathbb{C})$ and the homology bundle $H_{n-1} \to \Lambda \setminus \Sigma$ with fibre $H_{n-1}(V_{\lambda}, \mathbb{C})$. The bundles $H^{n-1}$ and $H_{n-1}$ are provided with the Gauss–Manin connection.

On the fibres of these bundles the group $G$ acts naturally. We consider the canonical decomposition (cf. [14]) of a representation of the group $G$ on the space $H_{n-1}(V_{\lambda}, \mathbb{C})$: $H_{n-1}(V_{\lambda}, \mathbb{C}) = \bigoplus \mathbb{C} H_{\lambda}(\lambda)$ (i.e., if $H_{n-1}(V_{\lambda}, \mathbb{C}) = U_1 \oplus \ldots \oplus U_k$ is the decomposition into the direct sum