In this paper we give a new definition of bicompactness of bitopological spaces, comparable with definitions of this kind known previously.

Several definitions of compactness of bitopological spaces have been given previously. Here we define the concept of bicompactness and investigate its properties.

1. Definition of Bicompactness. Let \((X,\mathcal{T}_1,\mathcal{T}_2)\) be a bitopological space. If \(\mathcal{U}_1 \setminus \{U^{1}_{k}\}_{k} \) respectively, \(\mathcal{U}_2 \setminus \{U^{2}_{k}\}_{k} \) are coverings of the spaces \((X,\mathcal{T}_1)\), respectively, \((X,\mathcal{T}_2)\), then the family \(\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \) is called a bicovering of the space \((X,\mathcal{T}_1,\mathcal{T}_2)\). We shall call a family from \(\mathcal{T}_1 \cup \mathcal{T}_2\) covering the set \(X\) a \(\mathcal{T}_1\mathcal{T}_2\)-open covering.

Definition 1. The bitopological space \((X,\mathcal{T}_1,\mathcal{T}_2)\) is called bicompact, if each bicovering of it contains a finite \(\mathcal{T}_1\mathcal{T}_2\)-open covering.

Example 1. By \(R^*_L, R^*_R\) and \(R^*_L, R^*_R\) we denote the real lines with the left and right topologies. In the union \(R^*_L \cup R^*_R\) we consider the topology of the bitopological sum. We denote the space obtained by \((R^*_L, R^*_R)\). This space is bicompact.

Several simple properties of bicompactness follow directly from the definition.

Theorem 1. Bicompactness is preserved under continuous maps.

Corollary 1. Bicompactness is a bitopological invariant.

Corollary 2. Let \((\prod X_k, \mathcal{T}_1, \mathcal{T}_2)\) be the bitopological product of the spaces \((X_k, \mathcal{T}_1^k, \mathcal{T}_2^k)\). If the product \((\prod X_k, \mathcal{T}_1, \mathcal{T}_2)\) is bicompact, then each of the spaces \((X_k, \mathcal{T}_1^k, \mathcal{T}_2^k)\) is also bicompact.

The converse of Corollary 2 does not hold in general as the following example shows.
Example 2. We denote by $X$ the set $[0,1] \subseteq \mathbb{R}$ of the real line. By $\mathcal{T}_d$, respectively, $\mathcal{T}_a$ we denote the discrete, respectively, antidiscrete topology. The spaces $(X, \mathcal{T}_d, \mathcal{T}_a)$ and $(X, \mathcal{T}_a, \mathcal{T}_d)$ are bicom pact, but the space $(X, \mathcal{T}_d, \mathcal{T}_a)$ is not bicom pact.

Proposition 1. If the space $X$ is compact in at least one of the topologies $\mathcal{T}_d$ or $\mathcal{T}_a$, then $(X, \mathcal{T}_d, \mathcal{T}_a)$ is a bicom pact space.

Theorem 2. A subset $C$ of the bicom pact space $(X, \mathcal{T}_d, \mathcal{T}_a)$, which is closed in each of the topologies $\mathcal{T}_d$ and $\mathcal{T}_a$ is bicom pact.

Proof. Let $\{U^d_i\}_{i \in I} \cup \{U^a_i\}_{i \in I}$ be an arbitrary bicovering of the subset $C$. Then there exist $\mathcal{T}_d$ -open sets $\{U^d_i\}_{i \in I}$ and $\mathcal{T}_a$-open sets $\{U^a_i\}_{i \in I}$, such that $U^d_i \cap C - U^d_i, U^a_i \cap C = U^a_i$. For the bicovering $\{(U^d_i)^{\mathcal{T}_d} \cup (V^a_i)^{\mathcal{T}_a}\}_{i \in I}$ of the space $X$, there exists a $\mathcal{T}_d$ -open covering $\{e_i\}_{i \in I} \cup \{U^d_i\}_{i \in I} \cup \{U^a_i\}_{i \in I}$. Consequently, $\{U^d_i\}_{i \in I} \cup \{U^a_i\}_{i \in I}$ is a $\mathcal{T}_d\mathcal{T}_a$-finite open covering of the subset $C$.

We note that bicom pactness does not transfer to subsets which are closed in only one of the topologies.

Example 3. We denote by $X$ the segment $[0,1]$ of the real line. Let $\mathcal{T}_d$ be the usual and $\mathcal{T}_a$ be the discrete topologies. The set $(0,1)$ is $\mathcal{T}_a$-closed in the space $(X, \mathcal{T}_d, \mathcal{T}_a)$ but is not bicom pact.

Proposition 2. Let $(X, \mathcal{T}_d, \mathcal{T}_a)$ be a bicom pact space. If $(X, \mathcal{T}_d)$ is compact, and $C \subseteq X$ is $\mathcal{T}_d$ -closed, then $(C, \mathcal{T}_d, \mathcal{T}_a)$ is bicom pact.

Proposition 3. If $C \subseteq (X, \mathcal{T}_d, \mathcal{T}_a)$ is a bicom pact set, then for each family $\{U^d_i\}_{i \in I} \cup \{U^a_i\}_{i \in I}$ of $\mathcal{T}_d$ -open and $\mathcal{T}_a$-open sets, $C \subseteq \bigcup U^d_i$ and $C \subseteq \bigcup U^a_i$, there exist $\{U^d_i\}_{i \in I}$ and $\{U^a_i\}_{i \in I}$ such that $C \subseteq \bigcup U^d_i \cup \bigcup U^a_i$.

Proposition 4. If $(X, \mathcal{T}_d, \mathcal{T}_a)$ is a strongly pairwise Hausdorff space, then each bicom pact subset can be separated from each point not belonging to it in the topology $\mathcal{T}$, where $\mathcal{T}$ is the supremum of the topologies $\mathcal{T}_d$ and $\mathcal{T}_a$.

Proof. We recall that $(X, \mathcal{T}_d, \mathcal{T}_a)$ is a pairwise Hausdorff space if for each pair of points $x, y \in X$ there exist a $\mathcal{T}_d$-open neighborhood $U^d$ of the point $x$ and a $\mathcal{T}_a$-open neighborhood $U^a$ of the point $y$, such that $U^d \cap U^a = \emptyset$ (cf. [1]). For an arbitrary point $x$ of a bicom pact set $C \subseteq (X, \mathcal{T}_d, \mathcal{T}_a)$ and a point $p \in X \setminus C$ there exist $\mathcal{T}_d$-open sets $U^d_x \subseteq X$ and $\mathcal{T}_a$-open sets $U^a_x \subseteq X$ such that $x \in U^d_x, p \in U^a_x \setminus U^d_x - \emptyset$ and $x \in U^a_x, p \in U^d_x \setminus U^a_x = \emptyset$. For the family $\{U^d_x\}_{x \in C} \subseteq C \subseteq \bigcup U^d_x, \bigcup U^a_x - \emptyset$ there exists a family $\{U^d_x\}_{x \in C} \subseteq \bigcup U^d_x \subseteq \bigcup U^a_x$, such that $C \subseteq (\bigcup U^d_x) \cup (\bigcup U^a_x)$.

The set $(\bigcup U^d_x) \cap (\bigcup U^a_x) = \emptyset$ is, as is $A$, $\mathcal{T}$-open and $C \subseteq A, p \in B, B \cap A = \emptyset$.

Proposition 5. Each bicom pact set of a bitopological strongly pairwise Hausdorff space $(X, \mathcal{T}_d, \mathcal{T}_a)$ is closed in the space $(X, \mathcal{T})$, where $\mathcal{T}$ is the supremum of the topologies $\mathcal{T}_d$ and $\mathcal{T}_a$.

Proof. For each point $p \in X \setminus C$, where $C$ is a bicom pact set in the space $(X, \mathcal{T}_d, \mathcal{T}_a)$, there exists a $\mathcal{T}$-open set $B_p \supseteq p$, $B_p \cap C = \emptyset$ from which it follows that $\bigcup_{p \in C} B_p = X \setminus C$ is a $\mathcal{T}$-open set.

Theorem 3. The union of a finite number of bicom pact sets is bicom pact.

Proof. It suffices to prove the theorem for the case of two sets. Let $C_1, C_2 \subseteq X, C_1 \cup C_2 = C$ be bicom pact subsets of the space $(X, \mathcal{T}_d, \mathcal{T}_a)$. If $\{U^d_i\}_{i \in I} \cup \{U^a_i\}_{i \in I}$ is an arbitrary bicovering of the set $C$, then for each of the bicoverings $\{U^d_i \cap C_1 \cup \{U^a_i \cap C_2\}_{i \in I}$ and $\{U^a_i \cap C_2 \cup \{U^d_i \cap C_1\}_{i \in I}$, there exists a finite bicovering $\{U^d_i \cap C_1 \cup U^a_i \cap C_2\}_{i \in I}$, respectively, $\{U^a_i \cap C_2 \cup U^d_i \cap C_1\}_{i \in I}$. The set $\{U^d_i\}_{i \in I} \cup \{U^a_i\}_{i \in I} \cup \{U^d_i\}_{i \in I} \cup \{U^a_i\}_{i \in I}$ is a finite $\mathcal{T}_d\mathcal{T}_a$-open covering of the set $C$. 

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