A method is proposed for the analysis of a periodic laminar boundary layer, refining the conventional methods of Lin, Rayleigh, and Hill and Stenning and providing a basis for the unification of those methods.

Derivation of the Fundamental System of Equations. The equations for a periodic laminar boundary layer have the form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + v \frac{\partial^2 u}{\partial y^2}$$

(1)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u = 0; v = 0 \text{ at } y = 0; u \to U(x, t) \text{ as } y \to \infty;$$

$$u = \phi(y, t) \text{ at } x = x_f.$$  

(2)

The velocity at the outer boundary of the boundary layer is given by the expression

$$U(x, t) = U_0(x) + W(x) \cos(\omega t).$$

The absence of a temporal boundary condition in the case of steady-state periodic motion renders it impossible, in principle, to solve the problem directly. This fact makes it necessary to adopt a specific representation of the time dependence of the functions $u$ and $v$.

We investigate the expansions of these functions in Fourier series, written in complex form:

$$u = u_0(x, y) + \text{Re} \sum_{s=1}^{\infty} u_s(x, y) \exp(i\omega t);$$

$$v = v_0(x, y) + \text{Re} \sum_{s=1}^{\infty} v_s(x, y) \exp(i\omega t).$$

(3)

Here $u_0$ and $v_0$ are unknown real functions, and $u_s$ and $v_s$ are unknown complex functions. The functions $u$ and $v$ can be represented by Fourier series, since they satisfy the sufficient conditions for expansion (periodicity with respect to time and differentiability at any point of the domain of definition); see the system (1) and the boundary conditions (2). Assuming sufficiently rapid convergence of the series (3), hereinafter we use segments thereof containing only two harmonics. We substitute these segments into the system (1), writing the velocity at the outer boundary of the boundary layer in the form $U(x, t) = U_0(x) + \text{Re}[W(x) \cdot \exp(i\omega t)]$. To take the operator Re for extraction of the real part outside the multiplication sign, we invoke the formula

$$\overline{\text{Re}z_1 \text{Re}z_2} = \frac{1}{2} \text{Re}(z_1 \bar{z}_2 + \bar{z}_1 z_2).$$

The overbar is used everywhere to denote the complex conjugate, and $z_1$ and $z_2$ denote arbitrary complex numbers. After the appropriate calculations, the first equation of the system (1) can be written

$$\sum_{p=0}^{4} \text{Re}\{N_p(u_0, u_1, u_2, v_0, v_1, v_2) \exp(i\omega t)\} = 0,$$

(4)
where $N_p$ denotes differential operators, the specific form of which will be given below. The latter sum consists of five terms in connection with the nonlinearity of the first equation of the system (1). From (4) we deduce

$$\sum_{p=0}^{4} [\text{Re} N_p \cos (\rho \omega t) - \text{Im} N_p \sin (\rho \omega t)] = 0.$$ 

Making use of the property of linear independence of the trigonometric functions, we obtain

$$\text{Re} N_0 = 0, \quad N_p = 0 \ (p = 1, \ldots, 4). \quad (5)$$

The second equation of the system (1) is transformed analogously. We go over to dimensionless variables in the system (5) according to the formulas

$$x = Lx', \ y = \delta y', \ z = \delta z', \ U_0 = U_m U_0, \ W = W_m W'.$$

(6)

For $u_0 = U_m U_0, \ v_0 = \delta U_m U_0, \ u_s = W_m U_s, \ v_s = \frac{\delta}{L} W_m V_s \ (s = 1,2).$

Here $L$ is a certain length scale, and $\delta = \sqrt{\nu L/U_{om}}$ is a quantity characterizing the thickness of the steady-flow region. The index $m$ refers to the maximum value of a function, and the prime to the dimensionless form. The scales $v_0$ and $v_S$ are determined by means of the equation of continuity.

A singular feature here is the introduction of a second scale with respect to the transverse coordinate, i.e., $\kappa = \sqrt{2 \nu \omega}$. It has been verified in several theoretical and experimental studies [2-4] that $\kappa$ is of the same order as the thickness of the region in which nonsteady motion in the boundary layer is concentrated. The introduction of $\kappa$ (which is aptly called the thickness of the vibrational boundary layer) is required in order to bring the quantities $v_s, \partial u_s/\partial y, \partial^2 u_s/\partial y^2$ to dimensionless form.

In dimensionless variables (we drop the prime from now on), the system (5) takes the form

$$1. \ u_0 \frac{\partial u_2}{\partial x} + v_0 \frac{\partial u_2}{\partial y} = U_0 \left \{ \frac{d U_0}{d x} + \frac{\partial^2 u_0}{\partial y^2} + \frac{1}{2} \beta \frac{d W}{d x} \right \} - \frac{1}{2} \beta \frac{\text{Re} \left \{ \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial x} + v_1 \frac{\partial u_1}{\partial x} + v_2 \frac{\partial u_2}{\partial y} \right \}}{\delta},$$

$$2. \ \frac{1}{2} \frac{\partial^2 u_1}{\partial \xi^2} - i u_1 = - i W + \alpha \frac{1}{\sqrt{2}} v_0 \frac{\partial u_1}{\partial \xi} + \alpha \left \{ \frac{d W}{d x} u_0 \right \} - \frac{1}{2} \beta \left \{ \frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial y} \right \} + \alpha \sqrt{2} v_1 \frac{\partial u_0}{\partial y},$$

$$3. \ \frac{1}{2} \frac{\partial^2 u_2}{\partial \xi^2} - 2 i u_2 = - \alpha \frac{1}{\sqrt{2}} v_0 \frac{\partial u_2}{\partial \xi} + \alpha \left \{ u_0 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_0}{\partial x} + \frac{1}{2} \beta \left \{ \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial \xi} \right \} \right \} + \alpha \sqrt{2} v_2 \frac{\partial u_0}{\partial y},$$

$$4. \ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_2}{\partial y} + v_2 \frac{\partial u_1}{\partial y} = 0,$$

$$5. \ u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial \xi} = 0,$$

$$6. \ \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0,$$

$$7,8. \ \frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} = 0 \ (s = 1, 2).$$

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