BIPOSITIVE PROJECTION OPERATORS IN A PARTIALLY ORDERED VECTOR SPACE

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For brevity, the vector space $E$ partially ordered with the aid of a linear semigroup $K$ such that $K \cap (-K) = \{0\}$, $E = K - K$, will be called the space $(E, K)$. By $P = P(E, K)$ we shall denote a class of linear homomorphisms $p: F \to E$ such that $p^2 = p; p: K \to K; p' = I - p: K \to K$. The elements of the class $P(E, K)$ will be called bipositive projection operators.

The study of the positive spectrum of a positive operator [1, 2] in the general case, when the operator is not indecomposable, can be greatly facilitated in a number of cases if the operator is preliminarily subjected to a quasitriangular decomposition with the aid of bipositive projection operators [3]. This circumstance was the main reason for defining the class $P$. In the sequel it was found that many properties of this class are not related to the topology of the space $(E, K)$, since they are algebraic properties. Therefore we shall confine ourselves at first to the case that the space $(E, K)$ is a simple vector space, and not a normed space. In this note we shall prove the first fundamental property of the class of bipositive projection operators ($P$ being a Boolean algebra).

Remarks. 1. To avoid misunderstanding, we intentionally refrain from calling the space $(E, K)$ a semi-ordered space, since this term is often used for vector configurations [4, 5].

2. In the class of all linear homomorphisms $\text{Hom}(E; E)$ it is possible to introduce a natural ordering as follows. Let $A, B \in \text{Hom}(E; E)$; we shall assume that $A \to B$ if and only if $A - B: K \to K$. Since $K \cap (-K) = \{0\}$, it follows that if $A \to B$, $A \to B$, then $A = B$.

Auxiliary Assertions. Let $(E; K)$ be a partially ordered space, and let $P = P(E, K)$ be bipositive projection operators.

1. Let $p \in P; x, y \in K$. Then:
   a) $x \geq px$;
   b) if $px \geq y$, then $py = y$.

2. Let $p, q \in P$. Then:
   a) if $pq = qp$, then $pq \in P$;
   b) the condition $q = pq$ is equivalent to the inequality $q \leq p$;
   c) if $p \geq q$, then $pq = qp$.

We shall prove only the Assertion 2c. Let $q \leq p$; it hence follows from Assertion 2b that $q = pq$. Next, let $s = qp$. We have $sq = (qp) q = q(pq) = q^2 = q$, hence $s - q = s - sq = sq'$. The operator $s$, being a product of two positive projection operators, is positive; therefore either $s - q = sq' \geq 0$, or $s \geq q$. Furthermore, $q = q(p + p') = qp + qp' = s + qp'$, hence $q \geq s \geq q$, and thus $s = q$.

THEOREM. The class of bipositive projection operators $P(E, K)$ acting in the space $(E, K)$ is a Boolean algebra under a natural ordering. If $p, q, r \in P(E, K)$, then:

a) $p \wedge q = pq = qp \in P$ and $p \wedge p' = 0$;

b) $p \vee q = p + q - pq \in P$ and $p \vee p' = 1$;
c) \((p \lor q) \land r = (p \land r) \lor (q \land r)\).

Let us preliminarily introduce the following concept. Let \(E_0\) be a linear subspace of \(E\), \(K_0 = K \cap E_0\).

If:

a) \(E_0 = K_0 - K_0\);

b) for any vector \(u \in K_0\) we have the inclusion \(\{z \in K \mid z \leq u\} \subseteq K_0\);

c) \(x_0 \in E_0\) and there exists a sup\(x_0 \in E_0\), then sup\(x_0 \in E_0\), we shall call in this case the space \((E_0, K_0)\) a component of the space \((E, K)\) \([4, 5]\).

Now we shall formulate an assertion that is also of intrinsic interest.

**LEMMA.** Let \(p, q \in P(E, K)\). Then the space \((pqE, pqK)\) will be a component of the space \((E, K)\).

Here

\[ pqE =qpE = \{x \in E \mid x = px = qx\}. \]

**Proof of Theorem.** Let \(r = pq\), \(s = qp\). Let us show that \(r = s\). According to the lemma we have \(rE = sE\). Hence if \(x \in E\), then \(rx \in sE\); according to (*) we hence obtain \(s(rx) = qp(rx) = q(rx) = rx\), i.e., \(sr = r\) and \(s - r = s - sr = s(I - r)\). But \(I - r = I - pq = I - p + p - pq = p' + pq' \geq 0\).

Thus \(s - r = s(I - r) = s(p' + pq') \geq 0\), or \(s \geq r\). By a similar reasoning we can prove on the basis of the condition \(sx \in rE\) that \(r \geq s\). Hence \(s = r\), and by virtue of Assertion 2a we have \(pq = qp \in P(E, K)\).

Let us show that \(p \land q = pq\). In fact, let \(m \in P\) and \(m \leq p, q\). Then \(m \leq q\) and \(pm \leq pq\). But since \(m \leq p\), it follows from Assertion 2b that \(m = pm \leq pq = qp = r\). On the other hand,

\[ pr = p(pq) = p^2 q = pq = r, \quad qr = q(qp) = r. \]

It therefore follows from Assertion 2b that \(r \leq p, q\), and hence \(p \land q = pq\).

Let us show that \(t = p + q - pq = p \lor q\). We have

\[ t = p + q - pq = p + pq' = q + p'q, \]

hence \(t \geq p, q\). Next,

\[ t^2 = (p + p'q)(p + p'q) = p^2 + p'qp + pp'q + p'q'q = p + p'q = t. \]

Here we used the commutativity of bipositive projection operators and the fact that \(pp' = 0\). Since \(t^2 = I\), \(t = I - t\) = \(I - p - p'q - p' - p'q = p'q' \geq 0\), we have proved that \(t\) is a bipositive projection operator and \(t \geq p, q\).

Now let \(m \in P\) and \(m \geq p, q\); it then follows from Assertions 2b and 2c that \(pm = p\) and \(mq = q\). Thus \(mt = mp + mpq' = p + qp' = t\); hence \(t \leq m\), and \(t = p \lor q\). Assertion c is a direct consequence of Assertions a and b.

Thus we have completed the proof of the theorem.

Remark. Bipositive projection operators occur in the study of \(K_\sigma\)-spaces \([4, 5]\). Their algebraic properties, however, are left aside in this case, and all the propositions related to \(P\) are derived from the condition of \(\sigma\)-completeness of a linear configuration. In \(K\)-linear manifolds which are not \(K_\sigma\)-spaces one does not deal with bipositive projection operators. On the other hand, with the aid of Stone's theorem \([6]\) it is easy to construct for any preassigned Boolean algebra \(\Sigma\), a space \((E, K)\) that is not a \(K_\sigma\)-space and in which the class \(P(E, K)\) is isomorphic to \(\Sigma\).

**LITERATURE CITED**