Higher approximations of ray asymptotics are investigated by the boundary-layer method. For sources that are naturally called a center of pressure and a center of rotation, direction diagrams are found for transverse and longitudinal waves, respectively, which are absent in a homogeneous medium.

This paper is devoted to the construction of the asymptotics as $\omega \to \infty$ of problems with point sources

$$I_{\omega} = \text{grad}(\nu(x)\text{div}u) - \text{rot}(\mu(x)\text{rot}u) + f(x)/\omega^2u = \tilde{F}(x).$$

The boundary-layer method is applied: the initial data for the ray formulas which are suitable not too near to the source are determined by matching with an inner expansion.

Two physically important problems are considered. Direction diagrams are found a) of the transverse wave in the case of a source

$$\tilde{F} = -4\pi(\text{grad} \delta(x-x')) + \tilde{p} \delta(x-x');$$

b) of the longitudinal wave from the source

$$\tilde{F} = -4\pi(\text{rot} \hat{e}_i \delta(x-x')) + \tilde{q} \delta(x-x');$$

where $\delta(x-x')$, $\delta$ is the delta function, and $\hat{e}_i$ is the unit vector along the $x_i$ axis.

Sources (0.2) and (0.3) model the natural problems of the excitation of an elastic medium by loads applied to a very small spherical cavity. In case a) this is uniform pressure, and then it is found that

$$\tilde{p} = \frac{\nu}{3} \left( \frac{\text{grad} \mu(x)}{\nu(x)} \right) \left. \frac{x-x'}{x-x'} \right|_{x-x'}. (0.4)$$

In case b) this is a rotational action of the type $\tilde{F} = \text{const}[\tilde{q},\hat{e}_i], \tilde{q}$ is the stress on the boundary of the cavity, $\hat{e}_i$ is the normal to it, and $[\cdot,\cdot]$ is the vector product. Here

$$\tilde{q} = \frac{\text{grad} \mu \cdot \text{grad} \mu_{x-i} \hat{e}_i}{\mu} \left. \frac{x-x'}{x-x'} \right|_{x-x'}. (0.5)$$

The transverse wave from source (0.2), (0.4) just as the longitudinal wave in the case (0.3), (0.5) is absent if the medium is homogeneous.

Similar questions for an equation differing from (0.1) by lower order terms and for sources with a less clear physical interpretation are solved by a closely related method in

The connection of sources (0.2), (0.4) and (0.3), (0.5) with the theory of a spherical radiator the author plans to treat elsewhere.
The procedure of constructing an inner expansion is presented below in a much more transparent manner than in [1]. The facts regarding the ray series and the matching needed to understand the computations are presented. A detailed consideration of higher approximations of the ray method and an algorithm for constructing all initial data for them (in the case of an arbitrary point source) are given in [1].

The paper begins with the study of the inner expansion in a scalar problem. Not only the technique developed here but concrete results as well are used in the theory of elasticity.

1. Inner Expansion in a Scalar Problem

We consider the equation

\[(\Delta x + \omega^2 n^2(x)) u = \delta(x - x'), \quad \omega \to \infty, \quad x, x' \in \mathbb{R}^3\]  

(\(\Delta x\) is the Laplace operator in the variable \(x\)) in a domain containing the point \(x'\) assuming that \(n^2(x)\) is infinitely differentiable and \(n(x) > 0\).

At some distance from \(x'\) (more precisely, for \(|x - x'| > \omega^{1+\varepsilon}, \varepsilon > 0\)) it makes sense to seek a formal asymptotic expansion of the function \(u\); see [2, Chap. 6].

1. Equations of the Boundary Layer. Near the source the inhomogeneity of the medium may be considered a small perturbation. To realize this idea we introduce the stretched boundary-layer coordinates

\[\tilde{x} = \omega(x - x'),\]  

we represent the function \(n^2(x)\) by its Taylor series

\[n^2(x) \sim N_0^2 + N_1(x - x') + N_2(x - x') + \ldots = N_0^2 + \omega^{4} N_0(x) + \ldots,\]  

where \(N_j, j > 0\) are homogeneous polynomials of degree \(j, N_0 = n(x) > 0\), and we seek \(u\) in the form

\[u(x, x', \omega) \sim \sum_{j \geq 0} \omega^{-j} V_j(x),\]  

assuming that \(\omega\) is contained in \(V_j\) only through \(\tilde{x}\).

We call expansion (1.4) an inner expansion.

Substitution of (1.2)-(1.4) into (1.1) and comparing coefficients of powers of \(\omega\) give [2, 3]

\[\mathcal{X} V_0 = \delta(x)\]  

(1.5) \[\mathcal{X} V_j = - \sum_{\gamma = 1}^{j} \mathcal{X}_\gamma V_{j-\gamma}, \quad j \geq 1;\]  

(1.6)

\[\mathcal{X} = \Delta + \gamma^2; \quad \mathcal{X}_\gamma = N_j(x), \quad \gamma = N_0.\]  

(1.7)

It is useful to consider the differential expression \(\mathcal{X} = \Delta + \gamma^2\) not only for \(\gamma = N_0\) but for complex \(\gamma\) as well.

In order that it be possible to match (1.4) with the ray series corresponding to waves issuing from \(x'\) [2], we impose the radiation condition (limiting absorption)

\[V_j(x) \to 0 \quad as \quad |x| \to \infty; \quad j \geq 0, \quad if \quad \Im \gamma > 0.\]  

(1.8)

Solutions of problem (1.5)-(1.6), (1.8) are naturally considered in the class \(\mathcal{S}(\mathbb{R}^3)\) of generalized functions of moderate growth. From [4] the uniqueness and existence of \(V_j, j = 0, 1, \ldots\) follow easily.