
COMBINATORIAL ANALYSIS (MATRIX PROBLEMS, ORDER THEORY)

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The survey is devoted to certain current problems in general combinatorial mathematics. The contemporary state-of-the-art of the theory of permanents, questions on the existence and the enumeration of matrices with nonnegative elements, and a number of problems connected with latin rectangles (enumeration, problem of completing a latin square, equidistant arrays) are examined. The main directions in combinatorial theory in connection with selection problems also are analyzed: matroid theory, transversals, extremal problems (coverings, depth of a matrix, Sperner families). Principal attention is paid to the papers reviewed in Referativnyi Zhurnal "Matematika" during 1975-1979.

The present survey is the first one on general combinatorial theory in the "Itogi Nauki" series. The number of studies in this area has increased rapidly during the past 20 years—a well-known fact. Combinatorial mathematics developed both in depth as well as in breadth. Together with the development of much effort on the construction of a stable foundation of combinatorial investigations, an immense amount of concrete new results was obtained both in the classical problems (for example, such as the solution of Euler's famous problem on orthogonal latin squares or the four-color problem in graph theory) as well as in the completely new previously unknown branches of the theory, arising both from its internal requirements and from the needs of numerous hard-to-see applications of discrete mathematics. All of this dooms to failure from the start any attempt to give a complete picture of the contemporary state-of-the-art of combinatorial analysis. The present survey should be looked upon as an attempt to throw light on only certain fundamentally important sides of general combinatorial theory that are very urgent at this time, as testified to, in particular, by the abundance of publications only partially reflected in the literature we have cited.

At the present time general combinatorial theory is divided into the following three parts: enumeration, configurations, and order theory. All these parts are present in the survey in a very condensed form. Thus, in the survey we do not touch upon the questions of the existence and construction of block designs, of projective planes and the generation on them of configurations (see [24, 106] in this regard). We leave aside also the many questions of enumerative theory as well as the diverse connections of combinatorial analysis with probability theory, referring the reader to the recently issued monographs [17, 18] and to the collection [13]. The algorithmic problems of combinatorial analysis (see [176]) also are not considered. All of these questions, in view of the extensiveness of the material, require separate surveys. We have explored in detail the combinatorial aspects of permanent theory and of matrices with nonnegative elements. In the part devoted to latin squares the main attention is on the problems of enumerating them, on the problem of completing a partial latin square, and on the questions of the existence of equidistant permutation arrays. Order problems, including matroid theory, fundamental to them, are examined relatively more completely (but, nevertheless, concisely by necessity). Here we have omitted certain topics, in particular, Ramsey's theory, combinatorial analysis of finite structures, and applications. In the survey we examine, first of all, the articles reviewed in Referativnyi Zhurnal "Matematika" during 1975-1979; however, with the aim of giving a clearer perspective we often include earlier papers. On the other hand, in the sections devoted to order prob-

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lems the availability of monographs and survey articles on the questions being examined (see [16, 21, 123, 168, 217]) permitted us, by starting off from them, to restrict the exposition more to the most recent years.

1. Permanents

The permanent of a matrix $A=(a_{ij})$, $i=1,...,n; j=1,...,m$, with elements from a commutative ring, is defined by the formula

$$\text{per } A = \sum_{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where the summation is over all injective maps $\sigma:[1,n] \rightarrow [1,m]$.

Main attention is paid to the case $m = n$.

One of the well-known problems for permanents is connected with the existence of a linear transformation $T$ of the space $M_n$ of $n \times n$-matrices with real elements, such that the identity

$$\text{per } A = \text{det } A \text{ for all } A \in M_n$$

is fulfilled (here $\text{det } A$ is the determinant of matrix $A$).

A negative solution to this problem was obtained in [152]; more precisely, it was proved that when $n > 2$ a $T$ such that identity (1) is fulfilled does not exist. Therefore, from then on either special types of transformations of space $M_n$ or transformations of subspaces of $M_n$ were analyzed. Thus, in [139] there was examined the space $H_n$ of symmetric matrices over a field $F$ being a subfield of the field of real numbers. It was proved that there does not exist a linear transformation of space $H_n$ with the property $\text{per } T(A) = c \text{det } A$, where $n \geq 3$, and $c$ is a given number.

Let a transformation $T$ of space $M_n$ consist in changing the sign of certain elements of matrix $A$.

We say that an $n$-th-order $(0, 1)$-matrix $A=(a_{ij})$ is convertible if a matrix $B=(b_{ij})$ exists such that $b_{ij}=\pm a_{ij}$ and $\text{per } A = \text{det } B$. Let $v(A)$ be the number of ones in matrix $A$.

It has been established in [96] that if $A$ is an $n$-th-order convertible matrix and $\text{per } A > 0$, then

$$v(A) \leq \frac{n^2 + 3n - 2}{2},$$

where equality is achieved if and only if permutation matrices $P$ and $Q$ exist such that $A = PT_nQ$, where $T = (t_{ij})$ is an $n$-th-order $(0, 1)$-matrix such that $t_{ij}=0 \iff 1 < i < j < n$. In particular, if $A$ is convertible and $n \geq 5$, then $v(A) < n(n-1)$ and equality is achieved if and only if $A$ has a zero row or a zero column.

Necessary and sufficient conditions for the convertibility of a square $(0, 1)$-matrix were found in [144]. Another question is connected with the description of linear transformations of space $M_n$ preserving the permanent.

It was shown in [148] that every linear transformation $T$ preserving the permanent has the form

$$T(X) = DPXQL \text{ or } T(X) = DPX'QL,$$

where $X \in M_n$, $P$ and $Q$ are permutation matrices; $D$ and $L$ are diagonal matrices; the prime denotes transposition. If as $M_n$ we take $M_n$, viz., the space of doubly stochastic matrices, then in [169] it was proved that in this case a permanent-preserving transformation $T$ has the form

$$T(X) = PXQ \text{ or } T(X) = PX'Q.$$

Linear transformations of space $M_n$, preserving not only the permanent but also the elementary symmetric functions of the roots of a matrix, were studied in [164]. In particular, it was established: if $T$ is a linear transformation of $M_n$, $\text{per } T(A) = \text{per } A$ and $E_2(T(A)) = E_2(A)$, where $E_2(A)$ is the second symmetric function of the roots of matrix $A$, then

$$T(A) = \pm D^{-1}P^{-1}A'P D \text{ or } T(A) = \pm D^{-1}P^{-1}A'PD,$$