On irrigated ground with poor natural drainage, artificial drainage is used to prevent secondary salinization and swamping. The calculation of the drainage parameters requires the solution of problems of saturated—unsaturated flow in the region between the drains [1-6]. The influence of the motion in the unsaturated region on the total flow of ground water is most important when the water-saturated region is thin and especially in the case of finely grained soil. If the capillary border intersects the surface of the ground and there is an alternation of the processes of infiltration, redistribution, and evaporation of moisture, rapid oscillations in the ground water level are possible. Under these conditions, calculation of the drainage parameters using hydraulic models [7], which do not take into account motion in the region of incomplete saturation, is made difficult by the need to specify the coefficient which measures the incompleteness of saturation, and also the rates of flow between the aeration region and the ground water. The present paper gives the mathematical formulation, an algorithm for numerical solution, and examples of calculations for a two-dimensional problem of unsteady joint flow of ground water and soil moisture.

1. Mathematical Formulation of the Problem

We consider the problem of two-dimensional unsteady flow with allowance for the region of incomplete saturation when the region of motion contains systematic horizontal drains (Fig. 1). It is assumed that the porous medium is incompressible, that the air pressure in the soil is equal to the atmospheric pressure, that the motion of the moisture takes place under the influence of capillary and gravitational forces, and that Darcy's law is valid in the case of complete and incomplete saturation of the porous medium. Allowance is made for the vertical layering of the soil. The dependences of the pressure $\psi$ and the coefficient of moisture conduction $k$ on the moisture $\theta$ are assumed to be single valued.

Under these assumptions, the determination of the basic characteristics of the saturated—unsaturated flow reduces to solution of the problem

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left[ k(\psi, z) \frac{\partial \psi}{\partial x} \right] + \frac{\partial}{\partial z} \left[ k(\psi, z) \frac{\partial \psi}{\partial z} \right] - f(\theta, z, t)$$

(1.1)

$$\psi(x, z, 0) = \psi^0(x, z), \quad 0 \leq x \leq l, \quad 0 \leq z \leq d$$

(1.2)

$$-k \left( \frac{\partial \psi}{\partial z} - 1 \right) = R(x, t), \quad 0 \leq x \leq l, \quad z=0$$

(1.3)

$$\frac{\partial \psi}{\partial z} = 1, \quad 0 \leq x \leq l, \quad z=d$$

(1.4)

$$\frac{\partial \psi}{\partial x} = 0, \quad x=0, \quad 0 \leq z \leq d$$

(1.5)

$$\frac{\partial \psi}{\partial x} = 0, \quad x=0, \quad b_1 \leq z \leq d$$

(1.6)

$$\psi = z - b, \quad x=0, \quad b_1 \leq z \leq b_2$$

(1.7)

On the section $0 \leq z < b$, the boundary conditions are specified in accordance with the type of drain.

If the drain is open ($b_1 = 0$), then

$$\psi = 0, \quad x=l, \quad b_1 \leq z < b$$

(1.8)
In the case of a closed drain completely filled with water \((b = b_1 = b_0)\), the condition (1.8) drops out. If the drain is not completely filled with water \((b > b_0 > b_1)\), then both conditions are satisfied. In the equations and boundary conditions, the \(z\) axis is directed vertically downward, \(b_1\) and \(b_2\) are, respectively, the coordinates along the \(z\) axis of the beginning and end of the drain, \(b\) marks the water level in the drain, and \(b_0\) marks the point at which the water table meets the side of the drain (Fig. 1a).

The conditions (1.3)-(1.6) and (1.9) reflect either the symmetry of the flow pattern or they specify the fluxes on the corresponding sections of the boundary. At the drain, the head \(\varphi = \psi - z = -b\) is assumed known. On the leaking section, the condition (1.8) is imposed. In determining the position of the point \(b_0\), allowance is made for the fact that \(\partial \psi / \partial x > 0\) for \(x = 1\), \(b_1 < b_0 < z < b\). The (ground) water table corresponds to the pressure isoline \(\psi = 0\), which is the boundary between the regions of complete and incomplete saturation.

To solve the problem (1.1)-(1.9), it is necessary to specify the form of the functions \(\Theta(\psi)\) and \(k(\Theta)\) for each lithologic layer of the aeration region, and also the function \(f\) when allowance is made for moisture removal by plant roots.

For the dependence \(k(\Theta)\), we have adopted here the formula [2]

\[
k = k_s \left[ \frac{(\Theta - \Theta_r)(\Theta - \Theta_0)}{(\Theta_s - \Theta_0)} \right]^n, \quad \Theta < \Theta_r; \quad k = 0, \quad \Theta \leq \Theta_0.
\]

The connection between the moisture and the pressure is determined by the formula [5]

\[
\Theta = \frac{\Theta_s - \Theta_r}{1 + (\psi/a)^n} + \Theta_r, \quad \psi < 0; \quad \Theta = \Theta_r, \quad \psi \geq 0
\]

where \(k_s\) and \(\Theta_s\) are, respectively, the filtration coefficient and the moisture in the case of complete saturation, and \(\Theta_r\) is the moisture corresponding to the bound water [2]. The parameters \(k_s\), \(\Theta_s\), \(\Theta_r\), \(n\), \(\Theta_0\), and \(a\) depend on the type of soil.

As in [6], the function \(f\) is taken in the form

\[
f = e(t)p(\Theta)q(z) / \int_0^{z_a} p(\Theta)q(z) dz, \quad 0 < z \leq z_a, \quad \text{if } \Theta \geq \Theta_r
\]

\[
q(z) = cz_a - z, \quad p(\Theta) = \begin{cases} 
1, & \Theta \geq \Theta_k \\
\frac{\Theta - \Theta_s}{\Theta_s - \Theta_0}, & \Theta_s < \Theta < \Theta_r \\
0, & \Theta \leq \Theta_*
\end{cases}
\]

where \(z_k(t)\) is the thickness of the root region, \(\Theta_*\) is the wilting moisture, \(\Theta_k\) is the critical moisture, and

\[
e(t) = \int_0^{z_a} f(\Theta) dz
\]