ON THE FORMULATION OF THE PROBLEM OF THE MOTION
OF A BODY IN WATER

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UDC 532.011.1

The selection of solutions describing steady irrotational flow of an ideal incompressible fluid over bodies is considered. The selection is based on restrictions that follow from the physical properties of a real fluid and from the presence of a boundary layer on the body. In particular, for any body one can specify a minimal Euler number below which flow without cavitation becomes physically impossible. In the limiting case of an Euler number equal to zero, only the Kirchhoff scheme is physically admissible, and the cavity section tends to a circle. An equation is derived for the limiting shapes of cavities at small cavitation numbers, and a comparison is made with known results.

In developed cavitation flows, the velocity field of the fluid depends on the geometry of the body and the position of the points of separation and further only on the difference between the pressures at infinity and on the cavity boundary, i.e., on the cavitation number \(\sigma = 2(p_0 - p_k)/\rho V_0^2\); it does not depend on the Euler number \(\text{Eu} = 2p_0/\rho V_0^2\) (\(\rho\) is the density of the fluid, \(p_k\) is the pressure in the cavity, and \(V_0\) and \(p_0\) are the velocity and pressure far from the body), since the equations that describe the flow of an incompressible fluid do not depend on an additive constant pressure. In such a formulation of the problem, regions with negative pressure can arise on the boundary of the body and within the flow [1].

In a real fluid at rest, the pressure cannot be negative,* since on the boundary of a solid body there is a boundary layer with motionless fluid. Therefore, as a restrictive condition in the mathematical formulation of the problem we shall have on the boundary of the body \(p \geq 0\) (this restriction need not hold on moving solid boundaries). A similar condition holds far from the body, where the fluid moves translationally and is at rest in a coordinate system attached to it; such a condition also holds on the cavity boundary.

1. We consider an irrotational flow with pressure \(p \geq 0\) on the boundaries of the region; then since the function \(p\) is hyperharmonic it follows [2] that \(p \geq 0\) in the complete region. Thus, positivity of the pressure in the complete flow region follows from the boundary conditions indicated above. This condition is not obvious directly from the properties of the fluid, since if the regions with \(p < 0\) are small or, conversely, the flow velocity is very large, the nucleating centers of cavitation bubbles may not succeed in developing sufficiently to modify the flow pattern significantly.

The restriction \(p \geq 0\) enables one to deduce some a priori properties of cavitation flows of physical significance.

As an example, we consider the limiting case of a steady flow with \(\text{Eu} = 0\) corresponding to small initial pressures or large velocities of the motion of the body. Then from these conditions and the Bernoulli integral we have \(V \leq V_0\). It follows from this that flows in an unbounded region with bounded surfaces of the cavities and solid boundaries have no physical meaning; far from the body and the cavity these flows are equivalent to a translational flow and a concentrated principal singularity (sink for the Efros scheme of dipole for the Riabouchinsky scheme, etc.). If at some point of space the velocities of the translational flow \(V_0\) and the velocity induced by this

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*We are not considering here specially purified water, in which negative pressures can be achieved.
singularity have opposite signs, then at the point symmetric with respect to the singularity the velocities will have the same sign. Thus, one can always find a point in the flow for which \( V > V_0 \), which contradicts the obtained restriction on the velocity. Also physically inadmissible are solutions with \( Eu = 0 \) with flow over a body in a tube or a periodic system of bodies, since the velocity must be greater than \( V_0 \) in the constructions.

Thus, in the case \( Eu = 0 \) we have simultaneously Kirchhoff's scheme and Brillouin's principle (1913) [2] on the maximum of the velocity on the cavity boundary, i.e., we have the conditions of Lavrent'ev's theorem [3], which states that in the two-dimensional and axisymmetric case the drag of a body is finite, the minimum of the drag being attained for bodies whose geometry coincides with the shape of the cavity in the case of flow over a plate or disk in accordance with Kirchhoff's scheme. This result is given in [2, 5].

We prove that the drag is finite in the general case. The proof is based on the almost obvious fact that sufficiently large flow tubes containing the body and an infinite cavity expand monotonically, and for \( p \gg 0 \) the external flow exerts a drag force on their surface. On the other hand, the internal flow far ahead of and far behind the body has the same velocity and does not change its momentum, so that the drag force must be applied only to the body.

To prove the existence of an expanding flow tube, we consider the flow potential \( \varphi_0 : \varphi_0 = V_0x + \varphi_1 \) as \( r_i \rightarrow \infty \), \( \text{grad } \varphi_1 \rightarrow 0 \), and on the boundary of the body and the cavity
\[
\frac{\partial \varphi_0}{\partial n} = 0, \quad \frac{\partial \varphi_1}{\partial n} = -V_0 \cos (nx)
\] (1.1)

On the boundary of the cavity \( V = V_0 \) and, therefore,
\[
\varphi_1 = \varphi_1(x_0) + \int_{x_0}^{x} \left[ V_0 dx - V_0 x_0 \right] = \varphi_1(x_0) - V_0 x_0 + V_0 \int_{x_0}^{x} \left[ 1 - \frac{1}{1 + (R')^2} \right] dx \approx \varphi_1(x_0) - V_0 x_0 + \frac{1}{2} \int_{x_0}^{x} (R')^2 dx
\] (1.2)

where \( x_0 \) is some section of the cavity, \( \tau \) is the coordinate along the streamline on the surface of the cavity, and \( R \) is the distance of the streamline on the boundary of the cavity from the \( x \) axis.

We represent \( \varphi_1 \) in the form of the potential of simple and double layers distributed over the surface \( \Sigma \) of the body and the cavity [3, 4]:
\[
\varphi_1 = \frac{1}{4\pi} \int_{\Sigma} \frac{1}{r} \frac{\partial \varphi_1}{\partial n} dS - \frac{1}{4\pi} \int_{\Sigma} \varphi_1 \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS
\] (1.3)

The normal is directed into the body and the cavity. Besides the condition on \( \Sigma \), we must also write down analogous integrals on an infinitely distant surface, but in the considered cases they are unimportant. For fixed dihedral angle \( \theta \) with apex on the \( x \) axis, we have
\[
dS = \frac{4}{\cos (nx)} dS_k \approx \frac{4}{R} dS_k
\] (1.4)

where \( dS_k \) is the increment of the cavity area within \( d\theta \). From (1.2) and (1.3) we find that for \( R_k^2 \leq R_k^2 \leq x/\ln^{1/2} x \) for points far from \( \Sigma \) the last integral in (1.3) is unimportant, since it converges and has a higher order of decrease than the first integral with respect to the parameters \( x_k \). There is no point in considering cavities that expand faster, since they are associated with infinite drag [4]. Then the estimate for \( \varphi_1 \) takes the following form when allowance is made for (1.1) and (1.4):
\[
\varphi_1 \approx -\frac{V_0}{4\pi} \int_{\Sigma} \frac{1}{r} dS_k
\] (1.5)

For \( R_k^2 < R_k^2 \leq x/(\ln x)^{1+\epsilon} \), where \( \epsilon > 0 \) is any small number, the integral in (1.5) converges, and for any finite section of \( \Sigma \) as \( r \rightarrow \infty \) the integral is equivalent to a source with intensity \( V_0 S_k/4\pi \) from which the flow in the region far from \( \Sigma \) has a monotonically expanding flow tube, which was what we needed to prove.

Now suppose that