INFLUENCE OF TRANSVERSE SURFACE CURVATURE ON BOUNDARY-LAYER SEPARATION

G. M. Bam-Zelikovich

INTRODUCTION

The influence of various factors on boundary-layer separation has been investigated in several papers. This influence is most conspicuously manifested in the variation of the boundary-layer separation criterion, i.e., of an appropriate dimensionless combination of the pressure gradient and other parameters, which attains a definite value at the separation cross section. An increase in the separation criterion signifies that, all other conditions being equal, the boundary layer can sustain a larger positive pressure gradient without separating, whereas a decrease in the criterion signifies that boundary-layer separation will set in at smaller pressure gradients.

The influence of compressibility on the separation criterion of a turbulent boundary layer has been investigated by Belyanin [1]; Zakharov [2] has studied the variation of the separation criterion for a plane turbulent boundary layer with heating or cooling of the wall; Vatazhin [3] has studied the influence of electromagnetic forces and distributed suction on the value of the separation criterion; and the author has investigated the effects of suction concentrated over a small length. An equation is given in [5] for the separation criterion of an incompressible turbulent boundary layer with allowance for transverse curvature of the surface. However, the derivation of that equation is not entirely correct. In the present article we give expressions describing the behavior of the separation criteria for a laminar and a turbulent boundary layer in a compressible gas with allowance for the influence of transverse curvature of the surface. This influence can prove significant when the thickness of the boundary layer is of the order of the radius of curvature of the flow surface.

§1. Consider the boundary layer formed by flow over an axisymmetric body or by flow through an axisymmetric duct. Let s be the coordinate measured along the generatrix, n the coordinate along the normal to the surface, and R the distance from the symmetry axis; see Fig. 1: a) \((\partial R/\partial n)\omega > 0\); b) \((\partial R/\partial n)\omega < 0\). Let us assume that there are no kinks in the generatrix. Then the boundary-layer equations for a compressible gas in axisymmetric flow have the form

\[
\frac{\partial R \rho u}{\partial s} + \frac{\partial R \rho v}{\partial n} = 0 \tag{1.1}
\]

\[
\rho u \frac{\partial u}{\partial s} + \rho v \frac{\partial u}{\partial n} + \frac{\partial p}{\partial s} = \frac{1}{R} \frac{\partial R T}{\partial n} \tag{1.2}
\]

\[
\rho c_v \left( u \frac{\partial T}{\partial s} + v \frac{\partial T}{\partial n} \right) = \frac{1}{R} \frac{\partial R}{\partial n} \frac{\partial T}{\partial n} - \frac{p}{R} \left( \frac{\partial R u}{\partial s} + \frac{\partial R v}{\partial n} \right) + \mu \left( \frac{\partial u}{\partial n} \right)^2 \tag{1.3}
\]

where \(\rho\) is the density, \(p\) the pressure, and \(T\) the temperature of the gas; \(u\) and \(v\) are the projections of the velocity onto the \(s\) and \(n\) directions; \(\mu\) and \(k\) are the viscosity and thermal conductivity coefficients; \(c_v\) is the specific heat at constant volume; and \(T\) is the frictional stress, which for a laminar boundary layer is given by the relation

\[
T = \mu \frac{\partial u}{\partial n} \tag{1.4}
\]

Since the boundary-layer thickness \(\delta\) has not been assumed small in comparison with the radius of curvature \(R_w\) of the cross section of the flow surface, \(R\) is a function of both \(s\) and \(n\).

The following conditions always hold at the wall:

\[ u = 0, \quad v = 0 \quad (n = 0) \tag{1.5} \]

It follows from the equation of motion (1.2) and its derivative with respect to \(n\) with allowance for the continuity equation (1.1) and the boundary conditions (1.5) that for \(n = 0\).
where the subscript \( w \) signifies that the quantities are evaluated for \( n = 0 \).

The following condition holds at the boundary-layer separation point:

\[
\frac{\partial u}{\partial n} = 0 \quad (n = 0) \tag{1.7}
\]

Differentiating Eqs. (1.2) twice and three times with respect to \( n \) and taking (1.1), (1.5), and (1.7) into account, we infer that at the separation point for \( n = 0 \)

\[
\left( \frac{\partial^2 R}{\partial n^2} \right)_w = 0, \quad \left( \frac{\partial^3 R}{\partial n^3} \right)_w = 0 \tag{1.8}
\]

The higher derivatives can no longer be expressed solely in terms of the pressure derivative \( \frac{dp}{ds} \).

In order to deduce the conditions on the velocity derivatives from expressions (1.6) and (1.8) for \( n = 0 \), it is necessary to know the derivatives of \( R \) and \( \mu \) with respect to \( n \). It is evident from Fig. 1 that

\[
R = R_w \pm \mu \cos \alpha = R_w (1 + \frac{A}{6}) \quad A = \pm \left( \frac{\delta}{R_w} \right) \cos \alpha \tag{1.9}
\]

where \( \alpha \) is the angle between the tangent to the generatrix and the symmetry axis, \( \delta \) is the boundary-layer thickness, the plus sign corresponds to outer flow over a body, and the minus sign, to flow in a duct. It follows from (1.9) that

\[
\frac{\partial R}{\partial n} = AR_w, \quad \frac{\partial^{k+1} R}{\partial n^k} = 0 \quad (k \geq 2) \tag{1.10}
\]

Inasmuch as \( \mu \) is a known function of the temperature, in order to compute the derivatives of \( \mu \) with respect to \( n \) it is necessary to know the derivatives of \( T \) with respect to \( n \).

We first consider the case of a thermally insulated wall. The temperature boundary condition is

\[
\frac{\partial T}{\partial n} = 0 \quad (n = 0) \tag{1.11}
\]

Inspecting Eq. (1.3) and its derivative with respect to \( n \) on the surface, i.e., for \( n = 0 \), and taking (1.5), (1.7), and (1.11) into account, we obtain

\[
\left( \frac{\partial^2 T}{\partial n^2} \right)_w = 0, \quad \left( \frac{\partial^3 T}{\partial n^3} \right)_w = 0 \tag{1.12}
\]

Since \( \mu \) is a function of \( T \) only, it follows from (1.11) and (1.12) that

\[
\left( \frac{\partial \mu}{\partial n} \right)_w = 0, \quad \left( \frac{\partial^2 \mu}{\partial n^2} \right)_w = 0, \quad \left( \frac{\partial^3 \mu}{\partial n^3} \right)_w = 0 \tag{1.13}
\]

Using expressions (1.9) for the derivatives of \( R \) and (1.13) for the derivatives of \( \mu \) for \( n = 0 \), we determine the derivatives of the velocity \( u \) at the wall from Eqs. (1.6) and (1.8):

\[
\begin{align*}
\left( \frac{\partial u}{\partial n} \right)_w &= \frac{1}{\mu w} \frac{dp}{ds}, \\
\left( \frac{\partial^2 u}{\partial n^2} \right)_w &= -\frac{A}{\mu w} \frac{dp}{ds}, \\
\left( \frac{\partial^3 u}{\partial n^3} \right)_w &= -\frac{12A^2}{\mu w} \frac{dp}{ds}
\end{align*} \tag{1.14}
\]

We seek the velocity profile at the separation point of a laminar boundary layer in the form of a sixth-degree polynomial in \( n \):

\[
u = \sum_{k=0}^{4} a_k n^k \tag{1.15}
\]

To determine the coefficients \( a_k \) we apply conditions (1.5), (1.7), and (1.14). We also require smooth matching of the unknown profile with the profile outside the boundary layer, i.e., we require that the following condition hold at the outer limit of the boundary layer:

\[
u = U, \quad \frac{\partial u}{\partial n} = 0 \quad (n = \delta) \tag{1.16}
\]

From this relation we obtain

\[
a_s = -(1/6\delta^3) \sum_{k=1}^{4} k a_k \delta^{k-1} \tag{1.16}
\]