Numerous methods have been developed to calculate the aerodynamic characteristics of wings of low aspect ratio in the case when there is flow separation from the wing edges. Among the methods based on direct solution of the three-dimensional Euler equations there are the method of discrete vortices [1, 2] and the panel method [3]. In addition, numerical and asymptotic methods [4, 5] based on the theory of slender bodies [6] are used. One of the most important shortcomings of this theory is the dependence of the flow pattern at a given section of the wing on only the upstream flow. The obtained solutions therefore contain no information about the influence of the trailing edge of the wing, on which, as is well known, the Chaplygin–Zhukovskii condition is satisfied. The aim of the present paper is to construct an asymptotic theory of higher approximation and a corresponding numerical method for calculating flow separation from wings of low aspect ratio in which this shortcoming is absent.

1. We consider steady symmetric flow with separation of an ideal incompressible fluid over a wing of zero thickness. Let the aspect ratio of the wing be $\lambda = O(1)$, and the angle of attack $\alpha = O(\lambda)$. As units of the velocity and the linear dimension we choose the velocity of the oncoming flow and the central chord of the wing. We introduce a rectangular coordinate system $x_1y_1z_1$ with origin at the tip of the wing, $x_1$ axis along the chord, $z_1$ axis along the span, and $y_1$ axis perpendicular to the axes $x_1$ and $z_1$. Let $\Sigma_i(x_1, y_1, z_1) = 0$ be the equation of a surface, where the subscripts $i = 1$ and $3$ correspond to the vortex sheets shed from the side edges, $i = 2$ corresponds to the wing surface, and $i = w$ to the wake behind the wing. Further, let $\Delta_{i,j}$ be the two-dimensional $(i = 2)$ or three-dimensional $(i = 3)$ Laplacian with respect to the variables $y_j, z_j$ and $x_j, y_j, z_j$, respectively; $u, v, w$ are the components of the velocity vector $V$, $p$ is the pressure, and $\{f\}$ is the discontinuity of $f$ on the transition through a singular surface.

The considered flow has a potential $\varphi(x_1, y_1, z_1, \alpha, \lambda)$, which satisfies the equation $\Delta_{3,1}\varphi = 0$, the boundary conditions on the wing $(\nabla \varphi \Sigma_1 = 0)$, and the boundary conditions on the vortex sheets: $|p|_{z_1}=0$, $(\nabla \varphi \Sigma_{2,3,w})=0$.

2. In the exterior region $\Omega_1$ with characteristic dimensions $O(1)$ and independent variables $x_1, y_1, z_1$ the wing is represented in the limit $\lambda \to 0$ by the interval $0 \leq x_1 \leq 1$ of the $x_1$ axis. In the first approximation

$$\varphi^{(1)} = x_1 + a y_1^{-1/2} x_1,$$

(2.1)

This solution does not satisfy the boundary conditions on the wing, and therefore we consider the interior region $\Omega_2$ with characteristic longitudinal dimension $O(1)$, transverse dimensions $O(\lambda)$, and independent variables $x_2 = x_1$, $y_2 = y_1/\lambda$, $z_2 = z_1/\lambda$, Moscow. Translated from Izvestiya Akademii Nauk SSSR, Mekhanika Zhidkosti i Gaza, No. 4, pp. 141-147, July-August, 1982. Original article submitted October 8, 1980.
In $\Omega_2$, we have

$$q^{(2)} = x^2 + \lambda q^{(2)}_1 \left( x_2, y_2, z_2, \frac{\alpha}{\lambda} \right), \quad \Delta_{x_2} q_1^{(2)} = 0, \quad q_1^{(2)} = \frac{\alpha}{\lambda} y_1 - \frac{1}{2} \frac{\alpha^2}{\lambda^2} x_2 \left( \sqrt{y_1^2 + z_2^2} \to \infty \right)$$

Thus, in $\Omega_2$ we have a two-dimensional unsteady problem of flow with separation over an expanding plate, the coordinate $x_2$ playing the part of the time. The solution of this problem does not contain singularities with respect to the variable $x_2$, and therefore to find the singular regions in the neighborhood of the leading and trailing parts of the wing it is necessary to construct a second approximation in $\Omega_2$.

We begin by considering the solution in the exterior region $\Omega_1$. In the limit $\lambda \to 0$, the wing in $\Omega_1$ degenerates into the interval $0 \leq x_1 \leq 1$, and the wake behind the wing into the ray $1 \leq x_1 < \infty$. The flow potential on the ray $[0, \infty)$ of the $x_1$ axis has a singularity with form determined by the principle of matching the solutions in $\Omega_1$ and $\Omega_2$:

$$q^{(1)} \sim x_1 + \alpha y_1 - \frac{\alpha^2}{2} x_1 + \lambda \varphi_1^{(1)} \quad \text{where } \varphi_1^{(1)} \text{ is the function determined by solving the interior problem in } \Omega_2.$$

It follows from (2.2) that

$$q^{(1)} = x_1 + \alpha y_1 - \frac{\alpha^2}{2} x_1 + \lambda \varphi_1^{(1)}, \quad \Delta_{x_1} \varphi_1^{(1)} = 0$$

where $\varphi_1^{(1)}$ has a given singularity on the ray $[0, \infty)$ determined from (2.2). Therefore, $\varphi_1^{(1)}$ is the potential corresponding to a dipole of given intensity:

$$\varphi_1^{(1)} = -\frac{y_1}{4\pi} \frac{\partial}{\partial z_1} \left\{ \int_0^\infty \frac{p_1(\xi)}{\xi} d\xi \right\}$$

Repeating the matching of the solutions in $\Omega_1$ and $\Omega_2$, we obtain a four-term expansion for $\varphi$ in $\Omega_2$:

$$q^{(2)} = x_2 + \lambda^2 \varphi_2^{(2)} + \lambda \ln \lambda \varphi_3^{(2)} + \lambda^2 \varphi_2^{(3)}$$

$$q_1^{(2)} = \frac{\alpha}{\lambda} y_2 - \frac{1}{2} \frac{\alpha^2}{\lambda^2} x_2, \quad \varphi_2^{(2)} = \frac{p_1''(x_2)}{4\pi} -$$

$$\frac{y_1}{8\pi} \left\{ \frac{p_1(0)}{x_2^2} + \frac{p_1''(0)}{1 - x_2} - \frac{p_1''(0)}{x_2} - p_1''(1) \ln(1 - x_2) - p_1''(0) \ln x_2 - p_1''(x_2) (1 + 2 \ln 2) + 2p_1'(x_2) \ln y_2 z_2 +$$

$$\int_0^\infty \frac{p_1''(\xi) \ln(\xi - x_2) \text{sign}(\xi - x_2) d\xi}{\xi} \right\} - \frac{1}{6} \frac{\alpha^2}{\lambda^2} y_2 \left( \sqrt{y_2^2 + z_2^2} \to \infty \right), 0 < x_2 < 1$$

$$q_2^{(2)} = \frac{y_2}{8\pi} \left\{ \frac{p_1(0)}{x_2^2} + \frac{p_1''(0)}{1 - x_2} - \frac{p_1''(0)}{x_2} - p_1''(1) \ln(1 - x_2) - p_1''(0) \ln x_2 - p_1''(x_2) (1 + 2 \ln 2) + 2p_1'(x_2) \ln y_2 z_2 +$$

$$\int_0^\infty \frac{p_1''(\xi) \ln(\xi - x_2) \text{sign}(\xi - x_2) d\xi}{\xi} \right\} - \frac{1}{6} \frac{\alpha^2}{\lambda^2} y_2 \left( \sqrt{y_2^2 + z_2^2} \to \infty \right), 0 < x_2 < 1$$

The functions $\varphi_2^{(2)}$ and $\varphi_3^{(2)}$ satisfy the equations $\Delta_{x_2} \varphi_2^{(2)} = 0, \Delta_{x_3} \varphi_3^{(2)} = -\partial^2 \varphi_2^{(2)}/\partial x_2^4$.

We note that the influence of the deformation of the wake on the flow characteristics in the neighborhood of the wing is $o(\lambda^4)$.

It follows from (2.4) that the expansion (2.3) is invalid in the neighborhood of the front part of the wing if at least one of the quantities $p_1(0), p_1'(0), p_1''(0)$ is nonzero. We consider the case $p_1(0) \neq 0$, which corresponds to a leading edge of the wing which is not sweptback. We introduce region $\Omega_3$ with characteristic dimensions $O(\lambda)$ and independent variables $x_3 = x_1/\lambda, y_3 = y_1/\lambda, z_3 = z_1/\lambda$. Matching of the solutions in $\Omega_2$ and $\Omega_3$ makes it possible to obtain the form of the asymptotic expansion for $\varphi$ in $\Omega_3$:

$$q^{(3)} = x_3 + \lambda^2 \varphi_1^{(3)}, \quad \Delta_{x_3} \varphi_1^{(3)} = 0$$

where $\varphi_1^{(3)}$ satisfies no-flow boundary conditions on the wing. Assuming separationless flow over the leading edge, we conclude that the vortex sheets are localized in the neighborhood of the end edges of the wing. Then in $\Omega_3$ the flow is separationless in the first approximation, and the problem for $\varphi_3^{(3)}$ is a problem in linear wing theory, the flow velocity being singular in the neighborhood of the end edge.