ADMISSIBILITY OF PITMAN ESTIMATORS
FOR A LOCATION PARAMETER

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1. Let us consider a sample \( X_1, X_2, \ldots, X_n \) of size \( n \) from a population having a probability density function \( f(x - \theta) \) with respect to Lebesgue measure. Here \( f(.) \) is a known function, and \( \theta \in \mathbb{R}^1 \) is an unknown location parameter, which is to be estimated.

An estimator \( \theta_n^* = \theta_n^* (X_1, \ldots, X_n) \) is called an admissible estimator of the parameter \( \theta \) (with respect to a quadratic loss function) if the following inequalities are impossible for any other estimator \( \theta_n \):

\[
E_{\theta_n} (\theta_n^* - \theta)^2 < E_{\theta_n} (\theta_n - \theta)^2, \quad \theta \in \mathbb{R}^1,
\]

\[
E_{\theta_n} (\theta_n^* - \theta_n)^2 > E_{\theta_n} (\theta_n - \theta_n)^2 \quad \text{for some } \theta_n. \quad \text{In other words, the estimator } \theta_n^* \text{ is admissible if it cannot be improved for all } \theta \text{ simultaneously.}
\]

The best invariant estimator for a location parameter is the Pitman estimator, which is defined as follows.* We put

\[
p_n(\theta) = \frac{\prod_{i=1}^{n} f(X_i - \theta)}{\int_{-\infty}^{\infty} \prod_{i=1}^{n} f(X_i - \omega) d\omega},
\]

and then define the Pitman estimator as

\[
\tilde{\theta}_n = \int \theta p_n(\theta) d\theta. \tag{1}
\]

Stein [2] has proved the following criterion for the admissibility of Pitman estimators.

THEOREM 1 (C. Stein). If

\[
E_{\theta_n} \left( \int \omega^p p_n(\omega) d\omega - \left( \int \omega p_n(\omega) d\omega \right)^p \right)^{\frac{1}{p}} < \infty,
\]

then the Pitman estimator \( \tilde{\theta}_n \) is admissible.

*The reader can find a similar description of the properties of the Pitman estimator in [1].

It follows from (2), in particular, that if \( \int |x|^\epsilon f(x) \, dx < \infty \), then the estimators \( \hat{\theta}_n \) are admissible for all \( n > 1 \). In the present note we seek to deduce from (2) admissibility criteria formulated directly in terms of the function \( f(x) \).

**THEOREM 2.** Let the following conditions hold:

\[
\int_{|x| > A} |x|^\epsilon f(x) \, dx = 0(\epsilon^a), \quad \epsilon > 0
\]  

(3)

\[
\int |x|^\epsilon f(x) \, dx = 0, \quad \delta > 0.
\]  

(4)

Then the Pitman estimators \( \hat{\theta}_n \) are admissible for all \( n \geq \frac{\delta}{\epsilon} + \frac{3}{\epsilon^a} \).

Condition (3) of the theorem is reasonable; if it does not hold, the estimator \( \hat{\theta}_n \) can become indeterminate for any \( n \). The role of condition (4) is less meaningful to the authors. It is, at best, a weak constraint. We also point out that for "normal" values of \( \epsilon, \delta, \epsilon > 1, \delta = 1 \) the upper bound for \( n \) is 12.

The proof of Theorem 2 is given in Sec. 3; some necessary supporting lemmas are given in Sec. 2. The proof of Theorem 2 entails basically the verification of inequality (2). Inasmuch as the left-hand side of (2) does not depend on \( \theta \), all of the ensuing arguments are pursued on the assumption that the "true" value of the parameter \( \theta \) is zero. With that understanding, we write \( P(x), E(x) \) instead of \( P(\cdot), E(\cdot) \).

2. Let us put

\[
Z_n(u) = \frac{\prod_j f(X_j - u)}{\prod_j f(x_j)}.
\]

Clearly,

\[
P_n(u) = \frac{Z_n(u)}{\sum Z_n(u) \, du}.
\]

The sample size \( n \) is assumed everywhere below to be a fixed number.

**LEMMA 1.** Under the conditions of the theorem, for all \( p < n \in \) - 1

\[
P \left[ \int Z_n(u) \, du > |A|^{-p} \right] = C_{p+1} A \frac{p+1}{p} \left( \frac{\epsilon A}{3} \right)^{-\frac{1}{p+1}} \]

(5)

Here \( p \) is an arbitrary positive number, and \( \epsilon \) is the number stipulated in condition (3). The symbol \( C_{p+1} \) always denotes constants.

**Proof.** For brevity we put \( \int Z_n(u) \, du = \eta \). For definiteness we assume that \( A > 1 \). We denote by \( \Gamma_j, j = \delta n, \) the event that exactly \( j \) of the \( n \) stochastic variables \( X_1, \ldots, X_n \) assume a value in the interval \( \Delta - \left[ \frac{A}{2}, \frac{5A}{2} \right) \). If we put \( \lambda = \frac{1}{n} \int f(x) \, dx \), then