


ACCURACY OF THE BOUSSINESQ APPROXIMATION FOR AN INCOMPRESSIBLE FLUID

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The corrections of first order to the eigenvalues and critical Rayleigh numbers obtained in the Boussinesq approximation are determined for convection in a fluid with zero compressibility. The ratio of the equilibrium difference of the densities to a mean density of the fluid is taken as the small parameter. The corrections are found by the methods of perturbation theory for self-adjoint operators. It is shown that in the class of problems with symmetry with respect to a horizontal plane the first-order corrections vanish. The restrictions on the system needed if the Boussinesq approximation is to be meaningful in the problem of the occurrence of convective instability are established.

1. The applicability of the Boussinesq approximation in various convection problems has been considered on a number of occasions. In [1, 2], isothermal compressibility of the fluid is taken into account. In [3, 4] it is shown that the Boussinesq approximation is the zeroth approximation to the exact equations in some small parameters. In [5], the approximation is studied in connection with convection in a viscoelastic medium.

The corrections to the critical numbers have been determined only in [2], the treatment differing from the Boussinesq approximation in allowing for isothermal compressibility; in the absence of such compressibility, the corrections vanish.

In the present paper, we consider convection in a fluid with zero isothermal compressibility, and we ignore the work of the pressure and internal dissipation in the heat balance equation. We determine the first-order corrections to the eigenvalues and critical values of the Rayleigh numbers in a small parameter which is the ratio of the equilibrium difference of the densities to some mean density. The problem is solved by decomposing the velocity field of the fluid into the main divergenceless field and a small perturbation due to the change in the density. The corrections are found by means of perturbation theory for self-adjoint operators.

It is shown that in problems with symmetry with respect to a horizontal plane the first-order corrections vanish. We establish the restrictions on the system under which it is meaningful to use the Boussinesq approximation in the problem of the occurrence of convective instability.

2. Suppose a Newtonian, heat-conducting, incompressible fluid (the density depends only on the temperature) occupies a region $G$ with fixed boundaries that are impermeable for the fluid. If we ignore the work of the pressure and internal dissipation in the heat balance equation, the motion of the fluid is described by the system of equations

$$\rho \frac{du}{dt} = -\nabla p + \text{div} S + \rho g,$$
$$\frac{d\theta}{dt} = \chi \Delta \theta,$$
$$\frac{d\rho}{dt} = -\rho \text{div} u.$$

Here, \( \rho \) is the density of the fluid, \( u \) is the velocity, \( \pi \) is the pressure, \( \theta \) is the absolute temperature, \( S \) is the viscous stress tensor, \( g \) is the acceleration due to gravity, \( \chi = \text{const} \) is the thermal diffusivity, \( \alpha = \text{const} \) is the coefficient of thermal expansion, \( \eta = \text{const} \) is the shear viscosity, \( \zeta = \text{const} \) is the bulk viscosity, \( I \) is the unit tensor, and \( \frac{d}{dt} \) denotes the total derivative with respect to the time.

We shall consider the flow under no-slip conditions on one part of the boundary, allowing slip on another. In both cases, one of the boundary conditions is vanishing of the velocity component normal to the boundary; in the first case, in addition, the tangential component is also zero, while in the second the corresponding tangential stress is zero.

The wall temperature is kept constant in time and such that the stationary temperature field corresponds to equilibrium state of the fluid.

Then in the initial equilibrium state \([6]\)

\[
\bar{V}\theta = -C\gamma, \quad C = \text{const} \tag{2.2}
\]

where \( \gamma \) is a unit vector in the opposite direction to \( \mathbf{g} \); we shall assume that the fluid is heated from below, i.e., \( C > 0 \).

In accordance with Lyapunov theory the convective stability and instability of the fluid are determined by the spectrum of the system of equations (2.1) linearized with respect to the deviations \( \{u, \theta\} \) from the initial equilibrium distributions \( \{0, \theta_i\} \).

Introducing the notation \( T = \theta - \theta_i, \ p = \pi - \pi_i \), we obtain the linearized system

\[
\rho_t \frac{\partial u}{\partial t} = -\nabla p + \text{div} S + \alpha \rho g T \gamma, \quad \frac{\partial T}{\partial t} = -u \cdot \nabla \theta_i + \chi \Delta T, \quad \text{div} u = \frac{\rho}{\rho_0} \chi \Delta T \tag{2.3}
\]

where \( \partial / \partial t \) is the time derivative at the given point of space, and \( g = |g|, \text{div} S = \eta \Delta u + (\eta/3 + \zeta) \text{div} u \).

Suppose \( T_o = \theta_i, \) \( T_o = \text{const} \), \( \nu = \eta/\rho_0, \ mu = (\eta/3 + \zeta)/\rho_0 \). Then, dividing the first equation by \( \rho_0 \), we obtain

\[
(1 + \alpha T_0) \frac{\partial u}{\partial t} = -\nabla \frac{p}{\rho_0} + \nu \Delta u + \mu \text{div} u + \alpha g T \gamma
\]

Following \([6]\), we go over to dimensionless quantities by means of the following characteristic quantities: the length \( \text{L} \), where \( \text{L} \) is the characteristic dimension of \( G \), the time \( \text{L}^2/\nu \), the velocity \( \chi/\text{L} \), the pressure \( \rho_0 \chi \text{L}^2 \), and the temperature \( \chi \text{L} \).

We obtain

\[
(1 + \varepsilon T_0) \frac{\partial u}{\partial t} = -\nabla p + \Delta u + \mu \text{div} u + RT \gamma, \quad P \frac{\partial T}{\partial t} = u \cdot \nabla T + \chi \Delta T, \quad \text{div} u = \frac{\varepsilon}{1 + \varepsilon T_0} \Delta T \tag{2.4}
\]

where we have introduced the Prandtl number \( P = \chi/\nu \), the Rayleigh number \( R = \alpha \chi \text{L}^2/(\nu \gamma) \), \( \mu_i = \mu/\nu \), and the ratio \( \varepsilon = \alpha \chi \text{L} \) of the initial difference of the densities to the mean density. The boundary condition for the temperature is \( T_1 = 0, \) and for the velocity on the one part of the wall \( u = 0 \) (no-slip condition) and on the other \( u \cdot n = 0 \) and \( n \times \text{Sn} = 0 \) (condition of slip), where \( n \) is the normal to the boundary.

For solutions proportional to \( \exp(\lambda t) \), the so-called normal perturbations, we obtain the eigenvalue problem

\[
\lambda (1 + \varepsilon T_0) u = -\nabla p + \Delta u + \mu \text{div} u + RT \gamma, \quad \lambda P T = u \cdot \nabla T + \chi \Delta T, \quad \text{div} u = \frac{\varepsilon}{1 + \varepsilon T_0} \Delta T \tag{2.5}
\]

Since the experiments are real for \( R > 0 \) \([6]\), the occurrence of instability corresponds to vanishing of \( \lambda \). The corresponding value of the Rayleigh number is called the critical value, and the velocity and temperature fields neutral perturbations.

It is obvious that if \( \varepsilon \rightarrow 0 \) for \( R = \text{const} \) then the system (2.5) tends to the system