Remark. In proving Theorem 4, we did not in fact use the other results of this paper, but rather used only the finite dimensional analog of the corollary to Theorem 2, which already follows directly from the results of [1]. We do not know if it is possible to weaken the condition that the modules $A/A^5$ and $B/B^5$ be finitely generated in the statement of Theorem 4.

LITERATURE CITED


ONE THEOREM OF COHN

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Let $F$ be the field of algebraic functions of one variable over the field of constants $k$, $v$ be a point of field $F/k$, and $A_v$ be the ring of functions not having poles outside point $v$. It is proved that $A_v$ is a $GE_2$-ring if and only if it coincides with the ring $k[X]$ of polynomials of one variable over field $k$.

For an associative ring $A$ we shall, as usual, denote the subgroup $GL_2(A)$ generated by elementary matrices by $E_2(A)$ and the subgroup generated by $E_2(A)$ and diagonal matrices by $GE_2(A)$. Following Cohn [2] we call $A$ a $GE_2$-ring if $GE_2(A) = GL_2(A)$. It is obvious that a commutative $A$ is a $GE_2$-ring if and only if $E_2(A) = SL_2(A)$. For any $a \in A$ we set $E(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$; it is easy to see that the matrices $E(a)$ form a generator system for $E_2(A)$.

If $F$ is the field of algebraic functions of one variable over the field $k$ of constants and $S$ is some finite set of points of field $F$, then by $A_S$ we shall denote a subring of field $F$ consisting of functions not having poles outside $S$. In the case when $S$ consists of one point we shall write $A_v$ instead of $A_{\{v\}}$. Cohn [2] proved that if $char k = 2$ and $deg v > 1$, then $A_v$ is not a $GE_2$-ring. Our aim is to prove the following result strengthening the above-mentioned Cohn theorem and showing, in particular, that the single $GE_2$-ring among rings of type $A_v$ is the ring $k[X]$ of polynomials of one variable over field $k$.

THEOREM. Assume that $E_2A_v$ is a normal divisor in $SL_2A_v$. Then $deg v = 1$, $F = k(X)$, and, consequently, $A_v \cong k[X]$.

Remarks. 1) If $card S > 1$, then the situation fundamentally differs from that which obtains when $card S = 1$. Thus, e.g., Vasershtein [1] proved that if $card S > 1$ and $k$ is a finite field, then $A_S$ is a $GE_2$-ring.

2) It can be shown that for $n \geq 3$, $E_n(A)$ is a normal divisor in $GL_n(A)$ for any commutative ring $A$.

Everywhere below we adhere to the following notation: $V$ is the set of points of field $F$; $v \in V$, an arbitrary (unless otherwise specified) point of field $F$; $A_v$, a subring of field $F$ consisting of functions not having poles outside $S$.

poles outside point \( v \); \( R_v \), a subring of \( F \) consisting of functions not having a pole at point \( v \); \( \mu_v \subseteq R_v \), an ideal of ring \( R_v \) consisting of functions vanishing at point \( v \) (\( \mu_v \) is the unique maximal ideal of local ring \( R_v \)); \( k_v = R_v/\mu_v \), residue field of point \( v \); \( \deg v = [k_v : k] \), degree of point \( v \); \( \pi_v : R_v \to k_v \), a canonic projection; \( \gamma \), order function corresponding to point \( v \).

The following lemma sums up the properties of ring \( A_v \) needed subsequently.

**LEMMA 1.** a) The quotient field of ring \( A_v \) coincides with \( F \).

b) \( \text{GL}_1(A_v) = k^* \) (the multiplicative group of field \( k \)).

c) If \( a \in A_v \), \( a \neq 0 \) then \( v_0(a) = 0 \).

da) If \( a \in A_v \), \( a \neq 0, -1 \) then \( v_0(a) = v_0(a^{-1}) \).

**Proof.** Let \( f \) be an arbitrary nonzero element of field \( F \); \( \mathcal{A}_v \), a divisor of the poles of element \( f \); \( \alpha, \gamma \), elements of ring \( A_v \). Let \( n \) be a positive integer such that \( n \cdot \deg v - d(\alpha)(\gamma) = 0 \) (here \( d(\alpha) \) is the degree of divisor \( \mathcal{A}_v \) of field \( F \)). By virtue of the Riemann –Roch theorem there exists \( \delta \neq 0 \) such that \( \delta(\alpha) = n \cdot \gamma(\delta(\alpha)) = 0 \). Then, obviously, \( a \in A_v \) and \( a \cdot f \in A_v \), which proves a). If \( a \in \text{GL}_1(A_v) \) then element \( a \) does not have poles and, consequently, is a constant. Assertion c) follows from the formula \( \sum_{\omega \in \mathcal{A} \setminus \mathcal{A}_v} v_\omega(a) \cdot d(\omega) = 0 \). If \( v_0(a) < 0 \), then \( v_0(1 + a) = \min(v_0(a), v_0(1)) = v_0(a) \). However, if \( a \in k^* \), then \( 1 \cdot a \in k^* \) and \( v_0(a) = v_0(1 + a) = 0 \).

Let \( \alpha \in \text{SL}_2(A_v) \); we shall say that \( \phi(a) \) is defined if \( \alpha_{ij} \neq 0 \) for all \( i \) and \( j \). In this case we set \( \phi(a) = v_0(d_{21}) - v_0(d_{12}) \). We shall say that \( \phi(\alpha) \) is defined if, first, the number \( \phi(a) \) is defined and equals zero and, second, \( v_0(d_{2i}) < 0 \) for some \( i \) and \( j \). In this case we set \( \psi(\alpha) = \psi_0(d_{2i}/d_{1i}) \).

**LEMMA 2.** Assume that \( \phi(a) \) (respectively, \( \psi(a) \)) is defined; then \( \phi(a) = v_0(d_{22}) - v_0(d_{12}) \) (respectively, \( \psi(a) = \psi_0(d_{2i}/d_{1i}) \)).

**Proof.** From Lemma 1d) we obtain: \( v_0(d_{1i}/d_{2i}) = v_0(1 + d_{1i}/d_{2i}) = v_0(d_{1i}) \) whence \( v_0(d_{1i}) - v_0(d_{12}) = v_0(d_{12} - d_{1i}) = v_0(d_{22}) - v_0(d_{2i}) \) and \( v_0(d_{2i}) = v_0(d_{2i}) \), therefore \( v_0(d_{2i}) < 0 \) for some \( i \) and \( j \). From this it obviously follows that \( v_0(d_{2i}/d_{1i}) < 0 \) and \( (d_{1i} - d_{2i}) \notin \mu_v \). Let \( \mathcal{B}_v = \mathcal{B}_v(d_{i1}) \); we shall say that \( \phi(b) \) is defined and equals zero and, second, \( v_0(d_{2i}) < 0 \) for some \( i \) and \( j \). In this case we set \( \psi(b) = \psi_0(d_{2i}/d_{1i}) \).

**LEMMA 3.** Let \( \alpha \in \text{SL}_2(A_v) \); \( \alpha = \beta \cdot E(\gamma) \); then: a) if \( \phi(a) \) and \( \phi(b) \) are defined, then \( \phi(\alpha) = \phi(b); \)
b) if \( \phi(a) \) and \( v_0(\beta_{ij}) < 0 \) for some \( i \) and \( j \), then \( \phi(b) \) is defined and \( \phi(b) = \phi(\alpha) \).

**Proof.** a) We have \( \beta_{12} = \alpha_{11}, \beta_{22} = \alpha_{22} \). Therefore, using Lemma 2, we obtain: \( \psi(b) = \psi_0(d_{22}) - \psi_0(d_{12}) = v_0(d_{22}) - v_0(d_{12}) = \psi_0(d_{22}) - \psi_0(d_{12}) \).

b) Let us show, first of all, that \( \phi(b) \) is defined. It is clear that \( \beta_{12} = \alpha_{11} \) and \( \beta_{22} = \alpha_{22} \) cannot equal zero. In addition, if \( \alpha_{11} = 0 \), then \( \beta_{12} : \beta_{22} = 1 \) and, consequently, \( \beta_{12} = 0 \) and \( \beta_{22} = k^* \). Since \( v_0(\beta_{2i}) = v_0(\beta_{1i}) \), we have \( \beta_{2i} \in k^* \), i.e., condition \( v_0(\beta_{ij}) < 0 \) is not fulfilled for any \( i \) and \( j \). From part a) it now follows that \( \phi(b) = 0 \) and, consequently, \( \psi(b) \) is defined. Finally, by virtue of Lemma 2, \( \psi(b) = \psi_0(d_{22}) - \psi_0(d_{12}) = \psi_0(d_{22}) - \psi_0(d_{12}) = \phi(\alpha) \).

**COROLLARY 4.** Let \( \alpha \in \text{E}_2(A_v) \); then:

a) if \( \phi(a) \) is defined, then \( \phi(\alpha) = \phi(a) \) for some \( a \in A_v \),

b) if \( \phi(a) \) is defined, then \( \phi(\alpha) = \phi(a) \).

**Proof.** a) Since \( \alpha \in \text{E}_2(A_v) \), there exist \( a_1, \ldots, a_n \in A_v \) such that \( \alpha = E(a_1) \cdot \ldots \cdot E(a_n) \). Let \( k \) be the smallest number for which \( \phi(E(a_1) \cdot \ldots \cdot E(a_n)) \) is not defined. We set \( \beta = \alpha \cdot E(a_1) \cdot \ldots \cdot E(a_n) \). Then