THE AXIOM OF DETERMINACY AND THE MODERN DEVELOPMENT OF DESCRIPTIVE SET THEORY

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This article contains a survey of modern investigations in descriptive set theory connected with the axiom of determinacy.

Introduction

Descriptive set theory, whose origins go back to the works of Borel, Baire, and Lebesgue at the turn of the century, developed into an independent subject during the twenties and thirties, occupying at that time a prominent place in mathematical research. Such world-famous scholars as P. S. Aleksandrov, L. V. Kantorovich, A. N. Kolmogorov, and M. A. Lavrent’ev spent time studying the descriptive theory (and achieved recognized results in it); and this field became one of the important areas in the mathematical activity of N. N. Luzin and P. S. Novikov.

It was largely through the efforts of Soviet mathematicians that such divisions of the descriptive theory as the theory of Borel sets, the theory of A-sets (also called analytic, or Suslin, sets), the theory of lattices, indices, and constituents, the theory of CA-sets and second-level projective sets, and the general theory of operations on sets (from which the theory of R-sets later evolved) were established and achieved, in the main, their finished form by the end of the 30’s.

All this research, now unified under the general name of classical descriptive set theory, is characterized from the modern point of view by the traditional concept, inherited from the theory of functions, of mathematical proofs as activity directed toward establishing the properties of objects having in some sense a real existence. One consequence of such an approach was the intuitive conviction of the researchers that every statement (or at least every “meaningful” statement) about “real” sets is either true—in which case it should be possible to prove it by a sufficiently great effort—or false, in which case it should be possible to refute it. The principal task of mathematicians is to find new techniques and methods of proof.

Such conceptions, typical of the majority of fields of mathematics, also “worked” well for a time in descriptive set theory, as long as the theory limited itself to such relatively “simple” sets as Borel sets or A-sets. However the situation changed completely when specialists in the descriptive theory turned to the study of the projective sets discovered by Luzin. While they had succeeded in establishing a theory rich in results about the first (lowest) level of projective sets formed by the Borel sets, A-sets, and CA-sets, yet only isolated essential results were obtained for sets of the second projective level, and the higher projective levels remained, in general, terra incognita. Essentially all that was known about them was that in each level there appear sets not found on the preceding levels. Moreover the reason why a definitive study was impossible lay not at all in deficiencies of technique. After the investigations of Novikov, R. Solovay, and others it became known (and Luzin had been convinced of this in the mid-20’s) that many important questions on projective sets of higher levels—and in some cases also second- and even first-level sets—do not in principle admit of a definite positive or negative answer on the basis of accepted mathematical methods and ways of reasoning.

Thus, Novikov [11] showed that no contradiction could be deduced from the assumption that there exists a Lebesgue nonmeasurable set of the second projective level. Later Solovay [69] established that it is also impossible to deduce a contradiction from the assumption that all projective sets of the real line are measurable. Thus the problem of the measurability of projective sets turns out to be undecidable. The same fate awaited the majority of the remaining open problems of the classical descriptive theory.

Naturally, such a situation led mathematicians working in the descriptive theory to seek new axioms.
not among the traditional postulates of classical mathematics, but admitting a more or less acceptable foundation and making it possible to obtain definite answers to the questions that are undecidable within the framework of the traditional approach. The first such supplementary axiom to be considered was Gödel's axiom of constructivity, whose principal applications to the problems of the descriptive theory were obtained by Novikov [11]. Definite interest was also aroused by the measurable cardinal axiom and the "sharps hypothesis" which is equivalent to it as far as applications to the descriptive theory are concerned. But the greatest attention and recognition among specialists in the descriptive theory over the last 10 or 15 years has been given to two axioms connected with infinite games: the axiom of determinacy (AD), and the axiom of projective determinacy (PD). It is to these axioms that the present article is devoted.

The popularity of this topic in contemporary research is attested by the mere fact that, besides a mass of journal articles, four volumes of the series Lecture Notes in Mathematics were devoted totally or to a significant degree to applications of determinacy. In chronological order, they were volumes 38, 37, 35, and 36. However, research in determinacy is reflected very little in Soviet publications: only §6 of Chapter 8 of the translated handbook [3] and Chapter 2 of the pamphlet [5] can be mentioned. This circumstance exerted a decisive influence on the choice of style for the present article. The author preferred to devote more space to the more important results, presenting them with proofs, rather than striving for a maximal coverage of all areas. The same method of exposition, we note, is adopted in the handbook mentioned above.

Now a few words about the structure of this survey. In the first section we present some necessary definitions and facts relating to projective sets. The following section §2 is an introduction to game theory and determinate sets. Then in §§3–7 we study the main applications of the axioms AD and PD to problems of the theory of projective sets connected with regularity properties, separability, uniformization, single- and countably-valued sets, and Borel and Suslin representations of projective sets. In the last section (§8) we present some results that are based on Martin's theorem on the determinacy of the Borel sets.

We close this introduction by pointing out the works from the bibliography which might be considered as an introductory course in the theory of determinacy. Article [59] (in particular, its first part) contains a survey of "early" research in determinacy. In the book [38] another field of applications of the axiom of determinacy is expounded—infinitary combinatorics. The fundamental monograph [58] included practically all the significant results of the descriptive theory connected with determinacy and obtained by the end of the 70's. The same could also be said of the articles [25, 34], except that here a narrower circle of questions is considered on a more popular level. To these we add the already-mentioned works [3, Ch. 8, §6] and [5]. In the works just enumerated, as in [23], certain philosophico-mathematical questions are also touched upon concerning the place of the hypothesis of determinacy among the accepted set-theoretical axioms.

§1. Projective Sets and the Projective Hierarchy

Descriptive set theory is concerned with sets located in certain definite spaces and consequently possessing an inherited external structure. Originally the research was limited, as a rule, to subsets of the real line R and the Euclidean spaces R^n. However, by the beginning of the 30's it was realized that for a variety of reasons it was more convenient to take as the basic space not the real line but a Baire space, leading to essential simplifications in certain important calculations.

Striving for geometric clarity, Luzin, in his Lectures [48], used a realization of Baire space in the form of the set J of irrational points of the line R. In modern works a different realization is used more often—the product of a countable number of copies of the set of natural numbers omega = {0, 1, 2, ...}—which facilitates the use of logical methods in the reasoning. Thus Baire space is taken to be the set N = omega^w of all omega-sequences of natural numbers endowed with the product topology. (The topology on omega is discrete.) Each point alpha in N can be represented in the form alpha = (a_0, a_1, a_2, ..., a_k, ...), where a_k = alpha(k) in omega for all k.

Together with Baire space, certain spaces derived from it of the form omega^l x N^m are usually considered; here l and m are natural numbers, not simultaneously zero. We shall call such spaces point spaces or

1Here is a list of articles presenting rather completely the trends mentioned: [4,§2], [14], [26], [34, par. 5.4], [50], [52], [67].