A class of complete binary code systems is presented in which one of the symbols of the alphabet is used in any code combination not more than once, in terms of prefix codes of this class and of the operations of inversion and substitution of codes.

1. A finite set $U$ of binary words is called a complete code system [1, 2] if the following two conditions hold: 1) any binary word can be factorized into a union of words in $U$ in a unique manner; 2) for any binary word $v$ Condition 1 does not hold for $U' = U \cup v$.

In this paper we consider a class $D$ of binary code systems of variable length. The class $D$ consists of complete binary code systems in which one of the symbols of the alphabet $\{0, 1\}$, say 1, is used not more than once in a code combination. Our purpose is to characterize fully the class $D$. Our result supports the well-known assumption that it is possible to describe a class of complete code systems in terms of prefix codes and of operations of code inversion and superposition [3]. The proof is based on a lemma that is of intrinsic interest.

2. Let $U = \{u_i | i = 1, \ldots, n + 1\} \in D$. Hence, the polynomial

$$Z_U(x, y) = \sum_{u \in U} x^{|u|_1} y^{|u|_0},$$

where $|u|_1$ is the number of occurrences of 1 in the word $u$, must have the form

$$x + c_1 xy^{k_1} + \ldots + c_{n-1} xy^{k_{n-1}} + y^n. \quad (1)$$

Without loss of generality we shall assume that $k_1 < k_2 < \ldots < k_{n-1}$. Using the completeness conditions [1], we can sharpen the values of the coefficients $c_i$ and of the exponents $k_i$. As is generally known,

$$x + x \sum_{j=1}^{n-1} c_j (1 - x)^{k_j} + (1 - x)^n \equiv 1. \quad (2)$$

Hence

$$c_{n-1} (-1)^{k_n-1} x^{k_{n-1}} + (-1)^{k_n} x^n \equiv 0,$$

and therefore

$$k_{n-1} + 1 = k \quad \text{and} \quad c_{n-1} = 1.$$  

By introducing these values into (2), we obtain

$$x + x \sum_{j=1}^{n-2} c_j (1 - x)^{k_j} + (1 - x)^{k_{n-1}} \equiv 1.$$  

In the same way we obtain $c_{n-2} = \ldots = c_1 = 1$, and $k_{n-1} = k-i$ for $i = 2, n-1$ and $k = n$. Thus, for $U \in D$ the polynomial $Z_U$ will be determined uniquely by the number of words in $U$, being expressed by

$$x(1 + y + y^2 + \ldots + y^{n-1}) + y^n. \quad (3)$$
and hence \( U = \{ 0^{a_1}10^{b_1} \mid a_1 + b_1 = i-1, i = 1, n \} \). Let \( A = \{ b_1 | a_1 = 0 \}, \ B = \{ b_1 | a_1 = 0 \} \) (i = 1, n).

It follows from the condition of unambiguous decoding that the only common element of \( A \) and \( B \) is the zero, whereas from the completeness condition it follows that any \( i = 0, n-1 \) can be represented (in a unique manner, for the reason mentioned above) in the form \( a + b \), where \( a \in A \) and \( b \in B \). Let \( V = \{ 0^{A_1}0^{B_1} \} \).

Let us show that \( U = \{ 0^{n} \} \cup V \). (It evidently suffices to prove that \( V \subseteq U \).) We shall use induction on the length of the words \( V \). For \( l = 1 \) our assertion holds, since \( 1 \in U \) by virtue of (3). Let us assume that \( 0^{a_1}10^{b_1} \in U \) for all \( a \in A, b \in B \) and \( a + b \leq l \), and let \( a + b = l + 1 \equiv n-1, a \in A, b \in B \). The word \( 0^{a_1}10^{b_1} \) must allow decoding (see [2]), and it is evident that it consists of three words of the system \( U \). Suppose that these words are \( u_1 = 0^{a_1}, u_2 = 0^{a_2}10^{b_2}, u_3 = 0^{a_3}1 \). But if \( b_1 \neq 0 \), then \( (10^{a_1}) = (10^{b_1}) \cdot \) \( (0^{a_2}) \) will be a nonidentity relation between words of \( U \), which is impossible. Similarly, if \( a_1 \neq 0 \), then \( (10^{b_1}) = (10^{a_2}) \cdot (0^{a_1}) \). Therefore \( a_1 = b_1 = 0 \) and \( u_2 = 0^{a_2}10^{b_1} \in U \), which completes the proof. Thus, we have for each \( U \) a factorization of the set

\[
N_n = \{ 0, 1, \ldots, n-1 \} = A + B,
\]

that satisfies the condition \( |A| \cdot |B| = n \); conversely, to any factorization (4) we can assign a code system \( U \) of class \( D \). Hence, it remains to characterize the factorization (4). This is a typical inverse problem of the additive theory of numbers. A series of such problems were considered in [4], where it is shown that usually it is possible to maximum information about the sets \( A \) and \( B \) if it is known that \( |A + B| \) is near the lower limit. As we shall see presently, the case where \( |A + B| \) coincides with the upper limit \( |A| \cdot |B| \) also sometimes gives us the possibility to obtain considerable information about the structure of the sets \( A \) and \( B \). Yet we shall consider a slightly more general problem.

3. Let \( N = \{ 0, 1, 2, \ldots \} \). To the factorization (4) we can assign two factorizations

\[
N = A' + B' = A + B',
\]

and it is clear that any natural number can be represented in the form \( a + b \) (\( a + b' \)) with \( a \in A', b \in B' \) (\( a \in A, b' \in B' \)) in a unique manner. We shall characterize all the factorizations \( N \) satisfying the latter condition, as well as the factorizations \( N_n \), specified by them. Since any factorization \( N_n \) is determined by a factorization \( N \), the problem will be solved. In referring below to factorizations, we shall always assume that the condition of uniqueness of representation of numbers by a sum is satisfied.

To the factorization \( N = A + B(\rho) \) we shall assign the sequence

\[
N_\rho = \{ n \mid N_\rho = A' + B', A' \subset A, B' \subset B \}.
\]

It is easy to see that if \( N_\rho = A' + B' \), then together with any \( a \in A \) (\( b \in B \)) the factorization \( A' + B' \) must contain all numbers \( a' < a \) (\( b' < b \)) belonging to \( A' \) (\( B' \)). Let us also note that \( N_\rho \subseteq A \cup B \). Suppose that \( M = \{ m_1 \} \) is a finite or infinite sequence of natural numbers, \( m_0 = 1, m_i > 1 \) (\( i = 1, 2, \ldots \)). Each natural number \( a \) will be written by us in the form \( (a_0, a_1, \ldots, a_i, \ldots) \), in view of the representation

\[
a = a_0m_0 + a_1m_1 + \ldots + a_im_i + \ldots,
\]

where \( 0 \leq a_i \leq m_{i+1}-1 \) and \( 0 \leq a_i \leq m_i < \infty \), if \( M \) is a finite sequence of \( k + 1 \) numbers. Let

\[
A_{\rho} = \{ a | a_0 = a_2 = \ldots = a_i = a_{i+1} = \ldots = 0 \},
\]

\[
A_1 = \{ a | a_0 = a_1 = a_2 = \ldots = a_{i+1} = \ldots \}.
\]

It is evident that \( N = A_{\rho} + A_1 \) is a factorization of \( N \); let us denote it by \( \rho_\rho \). It is also easy to see that

\[
N_{\rho_\rho} = \{ m_0m_1 \ldots m_{i+1} \mid 1 \leq p \leq m_{i+1}, i = 0, 1, \ldots \},
\]

and for \( \sigma \in N_{\rho_\rho} \) we have \( N_{\sigma} = A_{\rho} + A_1 \), where

\[
A'_{\sigma} = \{ a | a_0 \leq A_0, a < \sigma \}, A''_{\sigma} = \{ a | a \leq A_1, a < \sigma \}.
\]

**LEMMA.** For any factorization \( N = A + B(\rho) \) there exists a sequence \( M \) such that \( \rho = \rho_\rho \).

Let us denote by \( A_0 \) the set belonging to \( A \) and \( B \) in \( \rho \) that contains a 1, and denote the second set by \( A_1 \). It suffices to show that for any \( \nu = 1, 2, \ldots \), we have \( n_\nu = m_0m_1 \ldots m_\nu \) for some \( m_1, \ldots, m_\nu \), and for any \( j \leq \nu \) we have