ENDOMORPHICALLY CLOSED QUASIGROUPS

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We construct examples of an endomorphically closed loop and of an endomorphically closed TS-quasigroup that are not entropic.

A groupoid $G = \langle \sigma, \cdot \rangle$ is said to be a groupoid with additive endomorphisms, or an endomorphically closed groupoid if, for any endomorphisms $\alpha, \beta$ of $G$, the mapping

$$x (\alpha \cdot \beta) = x \alpha \cdot x \beta \quad (x \in G)$$

is also an endomorphism, and it is said to be entropic if the following identity holds on the groupoid:

$$xy \cdot zw = xz \cdot yw.$$ 

It is known that any entropic groupoid is endomorphically closed. By generalizing the concepts of endomorphic closure and of the entropic law to arbitrary algebras, Evans [1] has shown that a variety of algebras has the property of being endomorphically closed if and only if it is entropic. For individual algebras this is not true; Etherington [2] has constructed an endomorphically closed groupoid which is not entropic, V. P. Belkin [3] has constructed a semigroup with the same property. The present note is devoted to the construction of similar examples for loops and for TS-quasigroups.

By a quasigroup we mean an algebra of type $\langle 2, 2, 2 \rangle$ satisfying the well known identities $x = (xy)/y$, $x = y(x/y)$, $x = x/y$, $y = y(x/y)$. A quasigroup with an identity is called a loop.

We follow Evans [1] and say that a quasigroup is entropic if the following holds for any basic operations $f_1, f_2$ in $K$:

$$f_1(f_2(x, y), f_2(z, w)) = f_2(f_1(x, z), f_1(y, w));$$

we also say that a quasigroup is endomorphically closed if, for any endomorphism $\alpha, \beta$ of it and for any basic operation $f$, the mapping

$$x (f(\alpha, \beta)) = f(x \alpha, x \beta) \quad (x \in K)$$

is also an endomorphism.

Let $K$ be an idempotent quasigroup. Following [4] and [5] we adjoin the element 1 (1 $\in$ K) externally to the set $K$ and define the binary operations 1 (1 $\not\in$ K) on $L = K \cup \{1\}$ in the following way:

1) $x \circ y = x \cdot y$, $x \mathbin{\#} y = x/y$, $x \mathbin{\backslash} y = x \backslash y$, if $x, y \in K$, $x \neq y$;
2) $1 \circ x = x = 1$ for all $x \in L$;
3) $x \circ 1 = 1$ for all $x \in K$.

It is easily verified that the algebra $\langle L, \circ, \mathbin{\#}, \mathbin{\backslash} \rangle$ is a loop; we shall denote this loop by $L_K$.

THEOREM. Let $K$ be an idempotent quasigroup or finite or infinite order $n \geq 4$ with the property:

(i) any two distinct elements of $K$ generate $K$. 

TABLE 1

<table>
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<tr>
<th>1</th>
<th>a</th>
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Then for any endomorphism \( \varphi \) of the loop \( L_K \), other than the zero endomorphism \( \varepsilon : x \rightarrow 1 \), the bounded mapping \( \varphi = \varphi/K \) is an automorphism of \( K \).

Proof. In view of (i) \( L_K \) is generated by any elements \( a \) and \( b \) provided that \( a \neq b \), \( a \neq 1 \) and \( b \neq 1 \). If \( \varphi \) is an automorphism of \( L_K \) then the assertion is obvious. Let \( \varphi \) be a true endomorphism of \( L_K \), we shall show that then \( \varphi = \varepsilon \).

In fact, if there exist \( u \) and \( v \) such that \( u \neq v \) but that \( u \varphi = v \varphi \), then, by taking \( t = u \varphi v \), we obtain that \( t \neq 1 \) and that \( \varphi = 1 \). We take \( a \in L_K \) (\( a \) is different from \( t \) and \( 1 \)) and form the products: \( e = (a \circ t) \circ a \) and \( d = (t \circ a) \circ a \). It is easy to check that \( e \varphi = d \varphi = 1 \) and that \( e \neq 1 \) and \( d \neq 1 \).

We are going to show that \( e = d = t \). If, for example, \( d \neq t \), then \( t \) and \( d \) would generate the loop \( L_K \). Since \( t \varphi = d \varphi = 1 \), then \( \varphi \) would be the zero endomorphism. Hence \( e = d = t \), whence we have that \( a \circ t = t \circ a \) and that \( (a \circ t) \circ a = t \). Similarly \( a = (t \circ a) = t \). We shall next show that the elements \( z = (a \circ t) \circ t \) and \( w = t \circ (a \circ t) \) are equal to the element \( a \). In fact, if, for example, \( z \) were not equal to \( a \) then the elements \( r = z \circ a \) and \( s = z \circ (a \circ t) \) would be generators of \( L_K \), but \( r \varphi = 1 \) and \( s \varphi = 1 \).

Thus, we have proved that the elements \( 1, a, t, a \circ t \) form a loop of order four (see Table 1). Because the order of the loop \( L_K \) is greater than or equal to five, and because \( L_K \) is generated by the elements \( a, t \) we have arrived at a contradiction.

Thus, the theorem is proved.

Table 2 defines a quasigroup \( T \); it is not difficult to verify that \( T \) satisfies the hypotheses of the theorem. Let us show that the identity is the only automorphism of \( T \).

The following relations hold between the elements \( a, b \in T \): \( a = b(\hat{a}b), b = (\hat{b}a) a, b = a(b(ab)) \). If we operate on these qualities with the automorphism \( \varphi \) we obtain the system of equations

\[
x = y (y x), \quad y = (y x) x, \quad y = (y (y x)),
\]

where \( x = a \varphi, y = b \varphi \). The solutions of the first and second equations of the system (1) are respectively

\[
\begin{align*}
x & = a \varphi, \quad x = a \varphi, \quad x = a \varphi, \quad x = a \varphi, \quad x = a \varphi, \\
y & = b \varphi, \quad y = b \varphi, \quad y = b \varphi, \quad y = b \varphi, \quad y = b \varphi,
\end{align*}
\]

Their only common solution, other than \( x = a, y = b \), is the solution \( x = e, y = c \); but this is not a solution of the third equation of the system (1):

\[
e \neq e, \quad e (c (ec)) = e (ca) = ef = d,
\]

hence \( \varphi \) is the identity automorphism.

We form the loop \( L_T \); according to our theorem it only has two endomorphisms: the identity \( \varepsilon : x \rightarrow x \) and the zero endomorphism \( \varepsilon : x \rightarrow 1 \). In view of the identity \( x \circ x = 1 \) in \( L_T \) we have

\[
\begin{align*}
o & = \varepsilon \circ \varepsilon = \varepsilon / \varepsilon = \varepsilon, \quad e = a = a \neq a = a \neq e = a = a \neq e = a, \\
e & = e \circ o = o \circ e = e / o = o \neq e = o \neq e = o \neq e = o = e = o = e.
\end{align*}
\]

Hence, the loop \( L_T \) is endomorphically closed; however, the entropic law is not fulfilled in \( L_T \) because it is noncommutative.

We next construct an example of an endomorphically closed TS-quasigroup that does not satisfy the entropic law.

Let us recall that by a **TS-quasigroup** we mean a groupoid satisfying the identities

\[
x \cdot xy = yx \cdot x = y.
\]

The binary operation, with respect to which the identities (2) hold, is called the **TS-operation**. A **TS-quasigroup** can also be defined as a quasigroup in which the three operations \( \cdot, /, \backslash \) coincide.