For an arbitrary normed space \( X \) the set \( (X^{**})^\tau \) in \( X^{**} \) is introduced. It is proved that if \( X \) is a KN-lineal then \( X^{**} = (X^{**})^\tau \), where \( X^* \) is the Nakano dual to the Banach dual \( X^* \). By the same token \( X^* \) is not in any way related with any partial ordering in the KN-lineal \( X \).

It is well known that a partial ordering in a Banach lattice (a KB-lineal) \( X \) completely determines a topology in \( X \), but this topology contains relatively little information concerning the partial ordering. Some properties of the partial ordering are, however, uniquely determined by the Banach topology. A classical result in this direction is the following theorem due to T. Ogasavar: a KB-lineal \( X \) is a KB-space if and only if it is weakly sectionally complete, i.e., every weakly fundamental sequence in \( X \) is weakly convergent to an element of \( X \). Other results of the same type were proved by the author in [6]. The main aim of the present note is to prove that, for any KN-lineal \( X \), the space \( X^* \) of all completely linear functionals on the Banach dual \( X^* \) can be "extracted" from \( X^{**} \) without using any information concerning the partial ordering in \( X \) (Theorem 2). Other results obtained are related to the structure of spaces of regular and completely linear functionals (Theorems 1, 2, 3, 5, and 6) and to the properties of operators in these spaces (Theorem 4).

Terminology and Notation. The dual of a normed space \( X \) is denoted by \( X^* \). The symbol \( \tau \) denotes the canonical mapping of \( X \) into \( X^{**} \). The set of all continuous linear operators from the normed space \( X \) into the normed space \( Y \) is denoted by \( \mathcal{L}(X \rightarrow Y) \). If \( U \in \mathcal{L}(X \rightarrow Y) \), then we write \( U^* \) for the conjugate operator. If \( U \in \mathcal{L}(X \rightarrow Y) \) and \( U^{-1} \in \mathcal{L}(Y \rightarrow X) \) exists, then \( U \) is said to be an isomorphism of \( X \) onto \( Y \). The symbols \( m, l^p (p \geq 1), c_0, l^p = l^p(0,1) \) and \( M = L^\infty (0,1) \) have their usual significance ([4], pp. 67-74).

For terminology and notation concerning partially ordered spaces, we follow the monograph [2]. If \( X \) is a K-ideal, then \( X \) denotes the K-space of all regular functionals on \( X \), and \( X \) denotes the K-space of all completely linear functionals on \( X \). The space \( X \) is also said to be the Nakano dual to \( X \). A KN-lineal (or a KN-space) is a K-lineal (K-space) which is simultaneously a normed space with a monotonic norm. A KB-lineal is a KN-lineal complete in the norm. A KB-space is a KN-space \( X \) in which the following conditions hold:

(A) If \( x_n \) is a bounded sequence in \( X \), then \( \|x_n\| \rightarrow 0 \);

(B) if \( 0 \leq x_n \) is a sequence in \( X \) and \( \lim \|x_n\| < \infty \), then \( \sup x_n \in X \) exists.

Let \( X \) be an Archimidean ordered K-lineal and let \( u \in X_+ \). We denote by \( X(u) \) the set of all \( x \in X \) such that

\[ |x|_u = \inf \{ \lambda > 0 : |x| \leq \lambda u \} < \infty. \]

Following [1], we say that \( X \) is normal in itself if \( X(u) \) is complete in the norm \( \| \cdot \|_u \) for any \( u \in X_+ \). We recall that every K-space is normal in itself.

1. Lemma 1. Let \( X \) be a KB-lineal of bounded elements and let \( [f_t, T] \) be the family of elements in \( X^* \) such that \( 2T |f_t(x)| < \infty \) for any \( x \in X \). Then this family is summable in the norm topology of the space \( X^* \).

Proof. It can easily be shown that $\Sigma T | F(f_t) | < \infty$ for any $F \in X^{**}$. The space $X^*$ is a KB-space, and so it is weakly sectionally complete. It only remains to note that in a weakly sectionally complete Banach space $E$, the family of elements $\{S_\lambda, S\}$ is summable in the norm topology if $\Sigma S | f(x_\lambda) | < \infty$ for any $f \in E^*$. This fact is a direct consequence of the Orlicz-Pettis theorem ([3], pp. 104, 105).

**Lemma 2.** Let $X$ be an Archimedean K-lineal, normal in itself, and let $[f_t, T]$ be a point-bounded family in $\tilde{X}$, i.e., $\sup \{|f_t(x)| : t \in T\} < \infty$ for any $x \in X$. Then the family of modules $\{||f_t||, T\}$ is also point-ordered.

**Proof.** We fix any $u \in \tilde{X}$ and prove that

$$\sup \{|f_t|(u) : t \in T\} < \infty.$$ 

First consider the case $X = X(u)$. Since $X$ with the norm $\| \cdot \|_u$ is a Banach space, we have $X = X^*$ and $\sup \{|f_t| : t \in T\} = K < \infty$.

It follows that $|f_t|(u) \leq K$ for all $t \in T$. Now let $X$ be arbitrary. By the restriction of the functionals $f_t$ to $X(u)$ and application of the reasoning used above, we obtain the required result.

**Lemma 3.** Let $X$ be a K-lineal and let $[f_t, T]$ be a family of pairwise disjoint elements in $\tilde{X}$ such that there is a union $ST/f_t = f \in \tilde{X}$. Then

$$\sum_{t} |f_t(x)| < \infty, \sum_{t} f_t(x) = f(x)$$

for any $x \in X$.

This lemma follows simply from Theorem VIII.2.3 ([2], p. 233).

**Theorem 1.** Let $X$ be an Archimedean K-lineal, normal in itself, let $V$ be a normal subspace in $\tilde{X}$, and let $R$ be a component in $\tilde{X}$ generating $V$. Let $f$ be an additive homogeneous functional on $X$. The following assertions are equivalent:

(a) $f \in R$;
(b) there is a family $[f_t, T]$ in $V$ such that

$$\sum_{t} |f_t(x)| < \infty, \sum_{t} f_t(x) = f(x)$$

for any $x \in X$.

**Proof of (a) $\Rightarrow$ (b).** Choose a maximal family $[R_t, T]$ of nonempty pairwise disjoint elements of the $K$-space $R$ such that $f_t = PR_t f \in V$. Clearly $f \in \tilde{X}$, and it only remains to apply Lemma 3.

**Proof of (b) $\Rightarrow$ (a).** Firstly, Lemma 2 implies that $f \in \tilde{X}$. We start by considering the case in which $X$ is a KB-lineal of bounded elements. In this case it is sufficient to use Lemma 1 and the fact that components in $X^*$ are closed in the norm topology.

Now consider the general case, and assume that $f \notin R$. There is $g \in \tilde{X}$, such that $|g| \land |f| > 0$, but $g$ is disjoint from $R$. Let $u \in X_+$ be such that $(|g| \land |f|)(u) > 0$. Restriction of the functions $f, f_t$ to $X(u)$ and the employment of the preceding reasoning leads to a contradiction.

**Remark.** It is plain from the proof that the condition that the K-lineal $X$ be normal in itself is not essential for (a) $\Rightarrow$ (b). Simple examples show, however, that it is essential for (b) $\Rightarrow$ (a) to hold.

**Corollary.** Let $X$ be a K-space and let $F$ be an additive homogeneous functional on $\tilde{X}$. The following assertions are equivalent:

(a) $F \in \tilde{X}$;
(b) there is a family $[x_t, T]$ in $X$ such that

$$\sum_{t} |f(x_t)| < \infty, \sum_{t} f(x_t) = F(f)$$

for any $f \in \tilde{X}$. 579