THE SIEGEL-BRAJER THEOREM CONCERNING
PARAMETERS OF ALGEBRAIC NUMBER FIELDS

A. F. Lavrik

An upper bound is obtained for the product of the number of classes of ideals and the regulator of algebraic number fields.

1. We use the following notation: \( Q \) is an algebraic number field, \( K \) is a number field of degree \( n = r + 2t > 1 \) over \( Q \), \( r \) is the number of real conjugate fields in \( K \), \( t \) is the number of pairs of complex conjugate fields in \( K \), \( \pm d \) (\( d > 1 \)) is the discriminant, \( h \) is the number of classes of divisors, \( R \) is the regulator of \( K \), and \( q \) is the number of roots of unity contained in \( K \).

In the investigation of the asymptotic distribution of primes and of integral divisors of \( K \), the problem arises of finding an upper bound for \( hR/q \) depending on the other parameters of the field, i.e., the degree \( n \) and the discriminant \( d \). This problem also arises in the study of Artin's hypothesis concerning primes with a given primitive root.

The known Siegel-Brauer result \([1]\]
\[
\log hR \sim \log \sqrt{d}
\]  
concerns fields \( K \) normal over \( Q \) and satisfying the condition \( (n/\log d) \to 0 \). It follows from the proof given by E.S. Golod and I. R. Shafarevich \([2]\) of the existence of infinite unramified extensions of fields that the relation \( (1) \) is inapplicable to sequences of fields \( K \) of this type, since in this case \( n/\log d \) remains constant.

In this connection we have obtained the following result concerning the Siegel-Brauer theorem.

**Proposition.** For all fields \( K \)
\[
hR \leq c \sqrt{d} \ln^{-1} d,
\]  
where
\[
c = e^{3} \left( \frac{3}{\pi^3} \right)^{r} \pi^{-r/2} \left( 1 + \frac{1}{\ln d} \right)^{-1} q.
\]
In particular for \( d \geq 5 \)
\[
hR \leq q \sqrt{d} \ln^{-1} d.
\]

2. The inequality \( (2) \), like \( (1) \), is derived by using the abbreviated equation for the zeta-function of the field \( K \). We have the following general

**Lemma.** For \( Re \ s > \delta > 0 \) let \( \varphi(s) \) have an absolutely convergent Dirichlet series
\[
\varphi(s) = \sum_{n=1}^{\infty} a_{n} n^{-s}
\]
with real non-negative coefficients. Suppose that \( k \geq 1 \), the numbers \( A, \lambda, \alpha_{1}, \ldots, \alpha_{k} \) are positive, and
\[
\beta_{1} > 0, \ldots, \beta_{k} > 0.
\]

Let
\[ \Gamma_k(z) = \prod_{i=1}^{k} \Gamma(z, z \div \frac{3}{4}), \]
where \( \Gamma(\omega) \) is the gamma-function, satisfy the functional equation
\[ A^t \Gamma(z) \psi(\omega) = \lambda A^{-t} \Gamma(\delta - s) \psi(\delta - s), \] (3)
whose left side is a regular function everywhere in the complex plane except possibly at a finite number of poles.

Then for every real \( s > \delta \), we have, together with (3), the inequality
\[ \Gamma_k(s) \psi(s) > \sum_{\text{res}} \frac{1}{s - z}, \] (4)
where the summation is over all residues except that for \( z = s \).

**Proof.** Under the conditions of the lemma, Theorem 2 of [3] yields
\[ \sum_{\text{res}} \left| A^{s-t} \Gamma_k(\omega) \psi(\omega) \right| = \sum_{m=1}^{\infty} \sum_{m} \frac{\lambda}{m!} \gamma_k \left( \frac{m}{A} \right)^{\delta - s} \psi(\delta - s), \] (5)
where
\[ \gamma_k(\omega, X) = \frac{1}{2\pi i} \int_{\gamma} \Gamma_k(z) \psi(z) X^z \frac{dz}{z}. \] (6)

Here the integration is taken along a vertical straight line Re \( z = \delta \) to the right of all singularities of the integrand.

Consider the integral (6). Choosing \( \epsilon > 0 \) so that for \( \omega = s, \omega = \delta - s, X = m/A, \) and \( 1 \leq \nu \leq k \) we have
\[ a_\nu \lambda - a_\nu \omega - \beta_\nu > 0, \]
we express the \( \Gamma \)-factors of \( \Gamma_k(z) \) by their Euler integrals. After substitution in (6) and some simple transformations, we take the factor
\[ B = \frac{a_1 + \ldots + a_k}{a_1 \ldots a_k} \]
out from under the integral sign. We now change the order of integration (see [4], p. 150), and by using a known integral operator, we obtain
\[ \gamma_k(\omega, X) = B \prod_{\epsilon > 0} \prod_{\epsilon > 0} \exp \left( - X \frac{a_\nu}{\beta_\nu} \right) \frac{a_\nu - \Gamma(\delta - s)}{a_\nu - \beta_\nu} \] \[ \int d\beta_1 \ldots d\beta_k \]
where
\[ \alpha = a_1 + \ldots + a_k, \beta = \beta_1 + \ldots + \beta_k, \beta_\nu = a_\nu / a_\nu \]

Hence, under the conditions of the lemma,
\[ \gamma_k(\omega, X) > 0. \]

This result and (5) imply (4).

3. For \( \varphi(s) \) we now use the zeta-function \( \zeta_K(s) \) of the field \( K \). In this case we have
\[ a_1 = \ldots = a_r = \frac{1}{2}, a_{r+1} = \ldots = a_{k} = 1, \]
\[ \beta_1 = \ldots = \beta_k = 0, \]
\[ \lambda = 1, A = 2^{-1/2} \sqrt{d}, \]
and the number of integral divisors of \( K \) with norm \( m \).

The sum of the residues in (4) \( (z = 1, z = 0) \) is
\[ (2^{v}\pi^{v/2})^2 d^{v/2} \frac{\lambda}{q^v} \gamma_k(\delta - s), \]
and so
\[ hR \leq q^v \frac{\lambda}{q^v} \frac{\lambda}{q^v} \gamma_k(\delta - s). \] (7)