We consider the function space $\mathcal{B}_{p,q}^{l}(\Omega)$ of functions $f(x)$, defined on the domain $\Omega$ of a certain class and characterized by specific differential-difference properties in $L^p(\Omega)$. We prove a theorem on the embedding $\mathcal{B}_{p,q}^{l}(\Omega) \subset L^q(\Omega)$ in the case when $l = n/p - n/q > 0$ and its generalization for vector $l$, $p$, $q$.

In the present note we generalize results that were obtained earlier by V. P. Il'in [1]; they are based on his estimates [2] of integrals of potential type. Our attention was drawn to this problem by the one-dimensional case of the embedding which was established by P. L. Ul'yannov [3] using other methods.

In what follows we shall not use any properties of the modulus of smoothness other than the finiteness of the norm of a function belonging to the space $\mathcal{B}_{p,q}^{l}$. Let

$$p = (p_1, ..., p_n), \quad 1 \leq p_i \leq \infty,$$

$$\|\psi\|_p = \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left|\psi^{(\alpha)}(x_1, ..., x_n)\right|^{p_1} dx_1 \cdots dx_n\right)^{1/p_1},$$

$$\|\psi\|_{l,n} = \left(\int_{-\infty}^{\infty} \left|\psi^{(\alpha)}(x)\right|^{p} dx\right)^{1/p}, \quad \|\psi\|_{l,U} = \|\psi U\|_u,$$

where $\varphi U(x) = \varphi(x)$ for $x \in U$, $\varphi U(x) = 0$ when $x \not\in U$.

As is usual, when $p_i = \infty$, in the appropriate particular norm we take the essential supremum of the modulus of the function.

First of all we are going to derive an auxiliary estimate for an integral operator in terms of the above "mixed" norm.

**THEOREM 1.** Let

$$\Phi(x) = \int_{E_\infty} \frac{\varphi(x+y) y_\alpha}{|\varphi(y)|^{(1/q_1 - 1/p_1)\mu_1}} dy,$$

where

$$1 \leq p_i \leq q_i \leq \infty, \quad q_n = \infty, \quad x_i > 0, \quad \mu_i = x_i (1/p_i - 1/q_i) > 0,$$

$$i = 1, ..., n, \quad \alpha \geq 0, \quad \rho(y) = \sum_{i=1}^{n} |y_i|^{1/q_i}.$$

Let $1 \leq m \leq n - 1$, $p_m \geq p_k$, $q_m \leq q_k$ ($k = m + 1, ..., n - 1$) and, in addition, for $\varepsilon > 0$, $1 \leq p_\infty \leq q_\infty$, $1 \leq p_m < q_m < \infty$, and for $\varepsilon \geq 0$, $1 < p_\infty \leq q_\infty$, $1 \leq p_m < q_m < \infty$ or $1 = p_\infty$, $1 < p_m < q_m < \infty$.

Then

$$\|\Phi\|_p \leq C \|\psi\|_p. \quad (2)$$

**Proof.** We can assume that $m = n - 1$ since the general case is reduced to this case when we change the order of taking the norms with respect to $x_m, ..., x_{n-1}$; this change is possible in view of the generalized Minkowski inequality for integrals.

We use the following well-known one-dimensional inequality:
and apply this to \( \Phi(x) \), first with respect to \( x_1 \), then with respect to \( x_2 \), and so on.

Let us note that the following estimate for the norm of the kernel holds:

\[
\left\{ \sum_{i=1}^{n} \left| y_i \right|^{1/q_i} \left[ \left( 1 + \left| x_i - y_i \right|^{1/r_i} \right) \left( 1 + \left| x_i - y_i \right|^{1/r_i} \right) \right] \right\}^{1/r_i} \leq C_1 \left\{ \sum_{i=1}^{n} \left| y_i \right| + \sum_{i=1}^{n} \left| y_i \right|^{r_i/q_i} \left( 1 + \left| x_i - y_i \right|^{1/r_i} \right) \right\}^{1/r_i} \leq C_2 \left\{ \sum_{i=1}^{n} \left| y_i \right|^{1/r_i} \right\}^{2} \sum_{i=1}^{n} \left( 1 + \left| x_i - y_i \right|^{1/r_i} \right).
\]

It is obvious that the estimate is valid also when \( r_i = \infty \).

Thus, an application of (3) reduces the estimate (2) to the corresponding estimate in a space of lower dimension. After \( n - 2 \) such steps we come to the two-dimensional inequality (2) which has been established by V. P. Il'in [2].

Next we will cite the representation for a function in terms of an integral of differences that we obtained in [4] and will estimate the right-hand side of this representation with the help of (2).

Let \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be a vector with positive coordinates. By the horn \( \mathcal{R}(\sigma) \) we mean

\[
\mathcal{R}(\sigma) = \bigcup_{0 < h < \varepsilon} \{ x; 0 < a_i h^\sigma_i < x_i < b_i h^\sigma_i (i = 1, \ldots, n) \}.
\]

The following representation (see [4]) holds for almost all points \( x \):

\[
f(x) = \int_{E_n} H_n^{1-|f|} \Psi_n(y/H) f(x + y) dy + \int_{E_n} \sum_{i=1}^{n} \left( \int_{0}^{\varepsilon} H_n^{1-|f|} \Psi_n(y/H) \xi_i(t) dt \right) \Delta_n^{1-|f|} (x + y + t \varepsilon^i) dt dy dh.
\]

Here \( \Delta_n^{m_i}(t)f(x) \) is the difference of order \( m_i \) with a step in the direction of the \( i \)-th coordinate vector \( e_i; \sigma_i > 0; \)

\[
y/h^\sigma = (y_1/h^\sigma_1, \ldots, y_n/h^\sigma_n).
\]

The functions \( \Psi_n(x) \in C_0(E_n), \xi_i(t) \in C_0(E_i) \) and are concentrated in some cube from \( \{ x; 0 < x_i < 1, i = 1, \ldots, n \} \) and in some segment from \( (0, 1) \). We take their supports and \( \delta > 0 \) such that the actual integration on the right-hand side of (5) is carried out in the interior of a pre-assigned horn \( \mathcal{R}(\sigma) \), so that only the values of \( f(x) \) at the points of the horn \( x + \mathcal{R}(\sigma) \) enter into the representation (5) for this function. Let us stress that the parameter \( H > 0 \) can be arbitrary.

Let \( U \subset E_n \) be an open set; by \( U + \mathcal{R} = U + \mathcal{R}(\sigma) \) we mean the vector sum of the sets. For \( 1 \leq \theta_i \leq \infty \) we put

\[
\psi_i(x, t) = \int_{\mathbb{R}^{1-|f|}} \Delta_n^{1-|f|} (x + t \varepsilon^i) dt,
\]

if \( [x + t \varepsilon^i, x + (1 + \varepsilon \theta_i) t \varepsilon^i] \subset U + \mathcal{R}, 0 < t < H/\varepsilon^i \), and \( \psi_i(x, t) = 0 \) otherwise.

Let \( x \in U \). By substituting \( \psi_i(x, t) \) in the right-hand side of (5) and by estimating the integral with respect to \( h \), we obtain

\[
\left| D^f(x) \right| \leq C_1 \int_{U_n} H_n^{1-|f|} \left| \Psi_n^{(e)}(y/H) f(x + y) \right| dy + C_2 \sum_{i=1}^{n} \left( \int_{E_i} \left| y_i \right|^{1/\theta_i} + t^{1/\theta_i} \right) \left( \int_{E_i} \left| y_i \right|^{1/\theta_i} + t^{1/\theta_i} \right) dy dt.
\]

whence

\[
\left| D^f(x) \right| \leq \int_{E_n} H_n^{1-|f|} \left| \Psi_n^{(e)}(y/H) f(x + y) \right| dy + C_2 \sum_{i=1}^{n} \left( \int_{E_i} \left| y_i \right|^{1/\theta_i} + t^{1/\theta_i} \right) \left( \int_{E_i} \left| y_i \right|^{1/\theta_i} + t^{1/\theta_i} \right) dy dt.
\]

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