Exact estimates for the approximation of continuous functions by broken lines are found.

§1. Introduction

Let \([a, b]\) be some segment and let \(\omega(\delta)\) be the modulus of continuity on \([0, b-a]\). We let \(H_\omega\) represent the set functions \(f(x)\) continuous on \([a, b]\) such that

\[
\omega(\delta, f) \leq \omega(\delta) \quad \text{for all} \quad \delta \in [0, b-a].
\]

We let \(L_n(f; x)\) represent a broken line with nodes at points \((x_k, f(x_k))\) where \(x_k = a + \lfloor(b-a)/n\rfloor k, k = 0, 1, ..., n\) (\(n\) is fixed).

In this article we shall consider problems involved with approximating functions of class \(H_\omega\) by broken lines. In particular, we shall show that:

THEOREM 1. If \(\omega(\delta) \neq 0\) is the modulus of continuity on \([0, 1]\) and \(H_\omega\) is a corresponding class of functions on \([0, 1]\), then for any \(n = 1, 2, ...,\) we have

\[
\sup_{f \in H_\omega} \frac{\|f(x) - L_n(f; x)\|_c}{\omega(1/2n)} \leq \frac{3}{2} \sup_{\omega(\delta) \neq 0 \in H_\omega} \frac{\|f(x) - L_n(f; x)\|_c}{\omega(1/2n)} = \frac{3}{2}.
\]

Note that V. N. Malozemov [1] previously showed that:

THEOREM. If \(\omega(\delta) \neq 0\) is the convex modulus of continuity, then

\[
\sup_{f \in H_\omega} \frac{\|f(x) - L_n(f; x)\|_c}{\omega(1/2n)} = 1.
\]

§2. Auxiliary Statements.

LEMMA 1. Let \(\omega(\delta) \neq 0\) be the modulus of continuity on \([0, b]\) where \(b > 0\). Then for any \(0 \leq x \leq y \leq x + y = b\) we have the inequality

\[
x \omega(y) + y \omega(x) \leq \frac{3}{2} \omega\left(\frac{b}{2}\right).
\]

Proof. Consider the auxiliary function

\[
\omega^*(\delta) = \begin{cases} 
\omega(\delta) & \text{for} \quad \delta \in \left[0, \frac{b}{2}\right], \\
\omega\left(\frac{b}{2}\right) + \omega\left(\delta - \frac{b}{2}\right) & \text{for} \quad \delta \in \left[\frac{b}{2}, b\right].
\end{cases}
\]

Note that if \(\delta \in [b/2, b]\) we have \(\omega(\delta) \leq \omega(b/2) + \omega(\delta + [b/2]) = \omega^*(\delta)\); therefore, for any \(\delta \in [0, b]\), we have the inequality

\[
\omega(\delta) \leq \omega^*(\delta).
\]
We shall show that for the function $\omega^*(\delta)$, inequality (1) holds. Let $0 \leq x \leq y \leq x + y = b$. The following cases are possible:

a) $0 \leq x < b/4$,  
6) $b/4 \leq x < b/2$,  
$\alpha$) $x = b/2$.

It is clear that we always have

$$x\omega^*(y) + y\omega^*(x) = x\left[\omega(\frac{b}{2}) + \omega\left(y - \frac{b}{2}\right)\right] + (b - x)\omega(x).$$

$$= x\omega\left(\frac{b}{2}\right) + b\omega(x) + x\left[\omega\left(\frac{b}{2} - x\right) - \omega(x)\right] = A(x).$$

The last expression in case (a) is estimated as follows:

$$A(x) \leq \frac{b}{4} \omega\left(\frac{b}{2}\right) + b\omega(x) \leq \frac{b}{2} \omega\left(\frac{b}{2}\right) + \frac{b}{4} \omega\left(\frac{b}{2} - 2x\right) \leq \frac{3}{2} b\omega\left(\frac{b}{2}\right) = \frac{3}{2} b\omega^*\left(\frac{b}{2}\right).$$

For case (b) we have

$$A(x) \leq x\omega\left(\frac{b}{2}\right) + b\omega(x) \leq \frac{b}{2} \omega\left(\frac{b}{2}\right) + b\omega\left(\frac{b}{2}\right) = \frac{3}{2} b\omega^*\left(\frac{b}{2}\right).$$

In case (c) we have

$$x\omega^*(y) + y\omega^*(x) \leq \frac{3}{2} b\omega^*\left(\frac{b}{2}\right).$$

Therefore, for any $0 \leq x \leq y \leq x + y = b$, we have the inequality

$$x\omega^*(y) + y\omega^*(x) \leq \frac{3}{2} b\omega^*\left(\frac{b}{2}\right),$$

from which we obtain (1) by allowing for (2). Lemma 1 is proved.

**COROLLARY.** Let $\omega(\delta) \neq 0$ be the modulus of continuity on $[0, 1]$ and let $b \in [0, 1]$ be any number. Then

$$\sup_{x \leq y \leq b} [x\omega(y) + y\omega(x)] < \frac{3}{2} b\omega\left(\frac{b}{2}\right).$$

The statement follows from the lemma if we allow for the fact that $x\omega(y) + y\omega(x)$ is a continuous function on the closed set $\{(x, y): x + y = b\}$.

For $x\omega(y) + y\omega(x)$ we can also establish another estimate which, for particular moduli of continuity, is more exact than estimate (1). For example, we have:

**Lemma 2.** Let $\omega(\delta)$ be the modulus of continuity on $[0, b]$ where $b > 0$. Then for any $0 \leq x \leq y \leq x + y = b$ we have the inequality

$$x\omega(y) + y\omega(x) \leq \sum_{k=1}^{\infty} b \frac{b}{2^k} \omega\left(\frac{b}{2^k}\right).$$

**Proof.** We introduce the following notation:

the square $K_c = [0, c]^2$ and $M_c = \max_{(x, y) \in K_c} [x\omega(y) + y\omega(x)]$.

Let $0 \leq x \leq y \leq x + y = b$; then $x \leq b/2 \leq y$. We set $y = b/2 + z$. It is clear that if $(x, y) \in K_b$ and $x + y = b$, then $(x, z) \in K_{b/2}$ and $x + z = b/2$. On this basis we can write

$$M_b = \max_{(x, y) \in K_b} [x\omega(y) + y\omega(x)] = \max_{(x, y) \in K_{b/2}} \left[x\omega\left(\frac{b}{2} + z\right) + \frac{b}{2} \omega(x)\right] \leq \max_{(x, y) \in K_{b/2}} [x\omega(x) + z\omega(x)] + \max_{(x, y) \in K_{b/2}} \left[x\omega\left(\frac{b}{2}\right) + \frac{b}{2} \omega(x)\right] \leq M_{b/2} + b\omega\left(\frac{b}{2}\right).$$