A differencing scheme is introduced for a differential equation with a small parameter affecting the highest derivatives. In the case of an ordinary differential equation, the solution of the difference equation is shown to converge uniformly with respect to the small parameter.

Differencing methods of solution of simple boundary problems for the equation

\[ \nu \Delta u + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} + c(x) u = f(x) \]  

have been extensively worked out. The solution of the corresponding difference equations converges to the solution of the boundary problems for Eq. (1) provided that the step of the mesh tends to zero. However, in frequently encountered problems where the parameter \( \nu \) is extremely small, a mesh step is required which is extraordinarily, and virtually unfeasibly, small.

Hence it is helpful to have a homogeneous differencing scheme whose solution will be close to that of the original problem for all sufficiently small \( \nu \). It is known that replacing first derivatives \( \partial u/\partial x_i \) by central differences which gives good approximation (to second order) is not suitable for small \( \nu \), if the Laplacian operator is approximated in the usual way. This is true even for the homogeneous equation

\[ \nu u'' + u' = 0 \]  

with conditions

\[ u(0) = 0, \quad u(1) = 1 \]  

and

\[ u(x) = \frac{1 - \exp(-x/\nu)}{1 - \exp(-1/\nu)}, \]

while for the difference equation

\[ \nu u_{xx}^h + u_x^h = 0 \]

with the same conditions, the solution is

\[ u^h(kh) = \frac{1 - \left(\frac{2\nu - h}{2\nu + h}\right)^{kh}}{1 - \left(\frac{2\nu - h}{2\nu + h}\right)^{kh}}, \]

if \( \nu \neq h/2 \). Here and in the sequel is used the usual notation for differences:

- \( h \) is mesh size, \( Nh = 1 \), \( u_x(x) = (u(x + h) - u(x))/h \), \( u_x^h(x) = u_x(x - h) \), \( u_x = (u_x + u_x^h)/2 \).
It follows from the explicit form of solution that for \( \nu \ll h \) the solution of the difference equation has nothing in common with the solution of the boundary Problems (2), (3); for \( \nu = h/2 \) the differences solution does not even exist for the boundary problem.

The well-known method of getting rid of these sorts of difficulties consists in replacing first derivatives in Eq. (1) by one-sided differences (see, e.g. [1], [2]). The derivative \( \partial u/\partial x \) is approximated by the forward difference \( u_i^{+} \) if \( a_i(x) \) is positive and by the backward difference \( u_i^{-} \) in the contrary case. This method enables numerical solution of the boundary problem (at least in the regular case) for any positive \( \nu \) no matter how small. However, it, too, admits difficulties. In the first place, one-sided differences constitute approximation only to first order. Hence for values of \( \nu \) of size commensurable with \( h \), it would be more natural to regard the difference operator as approximating the differential operator

\[
\sum_{i=1}^{n} (\nu + \frac{h}{2} | a_i(x) |) \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} + c(x).
\]

We have here the so-called network viscosity which can exceed the viscosity described by the differential equation. Moreover, the difference equation poorly describes the boundary layer which shows up as diffused. For example, let us compare in the same way Problems (2), (3) and the difference equation

\[
u u_{xx}^{h} + u_{x}^{h} = 0
\]

under Condition (3). Here

\[
u u_{xx}^{h} = u_{x}^{h} = \frac{1 - \left(\frac{\nu}{\nu + h}\right)^{h}}{1 - \left(\frac{\nu}{\nu + h}\right)^{N}}.
\]

If we take as oriented boundary of the boundary layer for these problems points at which \( 1 - u(x) = e^{-4} \) (about 2% from the maximum overfall of the solution) then for \( \nu = h/4 \) the width of the true boundary layer \( \approx h \) whereas for the difference equation it comprises \( \approx 5h/2 \).

It follows from the explicit form of the solution also that the solution of the difference equation does not tend as \( h \to 0 \) to the solution of Problems (2), (3) uniformly in \( \nu \). In fact, for \( \nu = h \), \( u^{h}(h) = 2^{-1} (1 - 2^{-N})^{-1} \), and

\[
\lim_{h \to 0} [u(h) - u^{h}(h)] = 2^{-1} - e^{-1}.
\]

It would be expedient to construct a differencing scheme for Eq. (1) such that in the case of constant coefficients with solution of the differential equation of exponential type (and as specified on the boundary layer) this scheme would be a solution of the difference equation. It would also be desirable that outside the boundary layer and for finite values of \( 1/\nu \) the scheme would yield approximation not inferior to second order. It will simplify matters in the sequel to consider the case \( c(x) = 0 \) since all considerations are easily extended to the case \( c(x) \leq 0 \). We shall consider that the boundary of the region consists of pieces orthogonal to the coordinate axes and we write the differencing scheme as follows:

\[
\sum_{i=1}^{n} \gamma_i(x) u_{x_i}^{h} + \sum_{i=1}^{n} a_i(x) u_{x_1}^{h} = f(x),
\]

where the coefficients \( \gamma_i(x) \) are selected in accord with the considerations indicated above. In the case of constant coefficients \( a_i \), the homogeneous Eq. (1) has solution \( \exp(-a_i x_i/\nu) \). Substituting this solution into the homogeneous Eq. (4), we get

\[
\gamma_i = \frac{a_i}{2} \frac{\cosh a_i h}{2}\frac{1}{\nu}.
\]

Accordingly, it is natural to put

\[
\gamma_i(x) = \frac{a_i(x) h}{2} \frac{\cosh a_i(x) h}{2}\frac{1}{\nu}
\]

for the variable coefficients \( a_i(x) \).