ON ALGEBRAIC DIFFERENTIAL EQUATIONS
WITH KNOTTED TRAJECTORIES

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It is proven that a system of three autonomous differential equations with polynomials (of
degree no lower than three) for right sides can have a knotted curve as its solution.

In the investigation of the phase picture of a system of three differential equations
\[ \begin{align*}
x &= \mathcal{P}(x, y, z), \\
y &= \mathcal{Q}(x, y, z), \\
z &= \mathcal{R}(x, y, z)
\end{align*} \tag{1} \]
the question arises: what complex forms can be assumed by the components of the partition of phase space?
We can judge the complexity of the forms of these components by, in particular, the manner in which the
trajectories of system (1) are embedded in phase space. In this connection, it is of interest to explain
whether there exists a system (1) whose solution is knotted in the finite part of space (Fig. 1). The solution
to this problem is based on the following assertion.

**THEOREM 1.** There exist systems of differential equations of the form of (1), where \( \mathcal{P}, \mathcal{Q} \) and \( \mathcal{R} \)
are real polynomials of degree no lower than three admitting, for almost all values of the coefficients \( a_\alpha, b_\alpha, c_\alpha \), a solution of the form
\[ \begin{align*}
x &= \sum_{i=0}^{5} a_i x^i, \\
y &= \sum_{i=0}^{5} b_i x^i, \\
z &= \sum_{i=0}^{5} c_i x^i.
\end{align*} \tag{2} \]

**Proof.** To each collection of concrete values of the real coefficients \( a_\alpha, b_\alpha, c_\alpha \), and, consequently,
to each point of eighteen-dimensional Euclidean space \( E^{18} \):
\[ (a_0, a_1, a_2, a_3, a_4, a_5, b_0, b_1, b_2, b_3, b_4, b_5, c_0, c_1, c_2, c_3, c_4, c_5) \]
corresponds one curve of the form of \( L \).

In space \( E^{18} \) we introduce the following metric:
\[ \rho(A, A') = M \max_{0 \leq i \leq 5} \{|a_i - a'_i|, |b_i - b'_i|, |c_i - c'_i|\}, \]
where \( M > 0 \), and \( A \) and \( A' \) are points of space \( E^{18} \). The topology in \( E^{18} \) is
generated by this metric.

Let
\[ \begin{align*}
\mathcal{P} &= \sum_{i+j+k=0}^{5} P_{ijk} x^i y^j z^k, \\
\mathcal{Q} &= \sum_{i+j+k=0}^{5} Q_{ijk} x^i y^j z^k, \\
\mathcal{R} &= \sum_{i+j+k=0}^{5} R_{ijk} x^i y^j z^k.
\end{align*} \]

Fig. 1

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We seek the solution to system (1) in the form of (2). Then,

$$x^iy^jz^k = \sum_{\alpha=0}^{i+j+k} A_{ij}^{\alpha} t^\alpha,$$

(3)

where the $A_{ij}^{\alpha}$ are homogeneous polynomials of degree $i + j + k$ in the coefficients $a_\alpha$, $b_\alpha$, $c_\alpha$.

Substituting (2) and (3) into the first equation of system (1) and equating coefficients of like powers of $t$, we obtain the following system of sixteen linear inhomogeneous algebraic equations with the twelve unknowns $P_{ijk}$:

$$\sum_{i+j+k=0}^{3} A_{ij}^{\alpha} P_{ijk} = a_\alpha, \quad \sum_{i+j+k=0}^{3} A_{ij}^{\beta} P_{ijk} = 0,$$

(4)

$$\alpha = 1, 2, ..., 5, \quad \beta = 6, 7, ..., 15.$$

We note that, upon substitution of (2) and (3) into the two other equations of system (1) we obtain linear equations for finding the coefficients $Q_{ijk}$ and $R_{ijk}$ with exactly the same matrix as for system (4).

For system (4) to have at least one nontrivial solution, it suffices that the rank of its matrix

$$
\begin{array}{cccccccccccccccc}
4 & a_0 & b_0 & c_0 & A_{200} & A_{120} & A_{012} & A_{003} & A_{002} & A_{001} & A_{000} \\
0 & a_1 & b_1 & c_1 & A_{201} & A_{121} & A_{013} & A_{004} & A_{003} & A_{002} & A_{001} & A_{000} \\
0 & a_2 & b_2 & c_2 & A_{202} & A_{122} & A_{014} & A_{005} & A_{004} & A_{003} & A_{002} & A_{001} & A_{000} \\
0 & 0 & 0 & 0 & A_{203} & A_{123} & A_{015} & A_{006} & A_{005} & A_{004} & A_{003} & A_{002} & A_{001} & A_{000} \\
0 & 0 & 0 & 0 & A_{204} & A_{124} & A_{016} & A_{007} & A_{006} & A_{005} & A_{004} & A_{003} & A_{002} & A_{001} & A_{000} \\
0 & 0 & 0 & 0 & 0 & 0 & A_{007} & A_{006} & A_{005} & A_{004} & A_{003} & A_{002} & A_{001} & A_{000} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{008} & A_{007} & A_{006} & A_{005} & A_{004} & A_{003} & A_{002} & A_{001} & A_{000} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{009} & A_{008} & A_{007} & A_{006} & A_{005} & A_{004} & A_{003} & A_{002} & A_{001} & A_{000} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{010} & A_{010} & A_{010} & A_{010} & A_{010} & A_{010} & A_{010} & A_{010} & A_{010} & A_{010} \\
\end{array}
$$

(5)

equal sixteen.

Any minor of order sixteen of matrix (5) is either identically zero or is a homogeneous polynomial in the coefficients $a_\alpha$, $b_\alpha$, $c_\alpha$ of no lower than 33-rd degree. It is clear that if any minor of matrix (5) is non-zero at one point at least of space $E^{18}$ then it is certainly a polynomial which is not identically zero.

Consider some point

$$(0, a_1, 0, 0, 0, a_3, 0, 0, 0, 0, b_4, 0, 0, 0, 0, c_5, 0, 0)$$

of space $E^{18}$. At this point, minor $\Delta$ of order 16, obtained from matrix (5) by striking out the four columns consisting of elements of the form $A_{200}$, $A_{120}$, $A_{011}$, $A_{003}$, turns out to be equal to

$$\Delta = a_1^{\alpha} b_4^{\beta} c_5^{\gamma}.$$

Consequently, $\Delta \neq 0$, and $\Delta$ is a polynomial. Therefore, according to a well-known theorem on analytic functions, in any neighborhood of any point of space $E^{18}$ there exists a point at which $\Delta \neq 0$, and it is possible to solve a system (4) with such a minor. Solving system (4), we find the polynomial we sought

$$P = \sum_{i+j+k=0}^{3} P_{ijk} x^iy^jz^k,$$

which is the right side of a differential equation with solution of the form $L$. Theorem 1 is proven.

Let $\Gamma_{\infty}$ be the class of all smooth curves, with no points of self-intersection, of the form

$$L: x = x (t), \quad y = y (t), \quad z = z (t),$$

so embedded in $E^3$ that

$$\lim_{t \to \pm \infty} ||x(t)|| + ||y(t)|| + ||z(t)|| = \infty.$$

By compactifying $E^3$ by embedding the infinitely distant point, we can extend the definition of knot [1] to curves of class $\Gamma_{\infty}$. 

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