ON AN INVERSE PROBLEM FOR NON-SELFADJOINT DIFFERENCE OPERATORS

Yu. L. Kishakevich

We recover the coefficients in certain difference expressions in terms of a known generalized spectral function of Marchenko type.

In [1] we constructed a generalized spectral function of Marchenko type for a certain difference expression whose coefficients were elements of an arbitrary algebra with an identity element. In the present note we solve an inverse problem of spectral analysis relative to generalized spectral functions of Marchenko type for certain difference expressions with operator coefficients. The method we use is a natural generalization to the non-selfadjoint case of the process of "orthogonalization" of the operator polynomials $I, \lambda \cdot I, \lambda^2 \cdot I, \text{etc.}$ (see [2, 3]). We use the notation and definitions of [1].

1. Let $a = (A_{ij})$ be a matrix of order $n$ with elements from the algebra $\mathfrak{B}$. The set of all matrices of order $n$ over $\mathfrak{B}$ with the operations of matrix multiplication and addition forms a ring $\mathfrak{B}_n$. For each $i \neq j$ and for an arbitrary $C \in \mathfrak{B}$ let $b_{ij}(C)$ denote the matrix obtained from the unit matrix by replacing the element $A_{ij} = 0$ by $C$.

2. In this section $\mathfrak{B}$ denotes the algebra of linear operators acting in a separable Hilbert space $H$.

We solve the inverse problem of spectral analysis for the expression (C) (see [1]) that satisfies the following additional condition:

$$A_j = C_j; \quad A_j > 0; \quad j = 0, 1, 2, \ldots,$$

that is, for the expression

$$(LU)_j = A_j U_{j+1} + A_{j-1} U_{j-1} + B_j U_j$$

with the "boundary" condition

$$U_{-1} = 0.$$  \hfill (L)

THEOREM 2.1. For $R = (R_j)$ ($R_j \in \mathfrak{B}, j = 0, 1, 2, \ldots, R_0 = 1$) to be a generalized spectral function of the problem (K), (L) it is necessary and sufficient that the matrices

$$r_n = \begin{pmatrix} I & R_1 & \ldots & R_n \\ R_1 & R_2 & \ldots & R_{n+1} \\ \vdots & \ldots & \ldots & \ldots \\ R_n & R_{n+1} & \ldots & R_{2n} \end{pmatrix}$$

(2.2)

can be reduced, by using the generalized algorithm of Gauss, to triangular form with positive elements on the diagonal.

Proof. Necessity. Let

$$\Omega_n (\lambda) = \sum_{i=0}^{n} C_n i \lambda^i; \quad \Omega_n (\lambda) = \sum_{i=0}^{n} D_n i \lambda^i.$$

We write

By the equality (2.17) in [1] the following relations hold:
\begin{align}
\langle R | \lambda k \Omega_n (\lambda) \rangle &= \delta_{nk} \cdot \tilde{C}_{n,n}, \quad k \leq n, \\
\langle \lambda k \Omega_n (\lambda) | R \rangle &= \delta_{nk} \cdot D_{n,n}^1, \quad k \leq n,
\end{align}
these can be written as a system of linear equations in the unknowns $D_{n,0}, \ldots, D_{n,n}$ and $C_{n,0}, \ldots, C_{n,n}$ respectively:
\begin{align}
c_n x_n &= \tilde{a}_n, \\
r_n d_n &= \tilde{e}_n.
\end{align}
Since the condition of the theorem is automatically satisfied by $r_0 = I$, we assume that $r_1, r_2, \ldots, r_{n-1}$ also have the required property; by using the generalized algorithm of Gauss, we prove that $r_n$ can be reduced to triangular form with positive elements of the algebra $\mathfrak{B}$ on the diagonal.

By multiplying both sides of the system (2.7) by the corresponding matrices $b_{ij}(C)$ and by taking account of the inductive hypothesis, we transform the system (2.7) into the triangular form
\begin{align}
\begin{pmatrix}
I & R_1 & \cdots & R_n \\
0 & R_1 & \cdots & R_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R_n'
\end{pmatrix}
\begin{pmatrix}
D_{n,0} \\
D_{n,1} \\
\vdots \\
D_{n,n}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\end{align}
It follows from here that $R_{2n} = C_{n,n}^{-1} D_{n,n}^{-1}$. On account of the equalities (2.5) and (2.6) in [1], we obtain:
\begin{align}
R_{2n} &= \mathfrak{A}_n \mathfrak{A}_n^*, \\
\mathfrak{A}_n &= A_0 A_1 \cdots A_{n-1}.
\end{align}
It follows from the relations (2.9) that $R_{2n} > 0$, and the necessity of the condition is thereby proved.

**Sufficiency.** We prove this in two stages.

1) Let $R = \{ R_j \} (R_j \in \mathfrak{B}, R_0 = I, j = 0, 1, \ldots)$, and suppose that the matrices $r_n$ (see (2.3)) can be reduced to triangular form with positive elements from $\mathfrak{B}$ on the diagonal. We use this sequence to define a right and a left $\mathfrak{B}$-mapping of $R$ by the formulas:
\begin{align}
\langle R | P (\lambda) \rangle &= \sum_{i=0}^{n} R_i P_i, \\
\langle P (\lambda) | R \rangle &= \sum_{i=0}^{n} P_i R_i, \\
\langle P (\lambda) \rangle &= \sum_{i=0}^{n} P_i R_i \subseteq \mathfrak{B}^n.
\end{align}
We then construct two systems of polynomials $\Omega_n (\lambda), \tilde{\Omega}_n (\lambda)$ of degree $n$ whose coefficients are elements of $\mathfrak{B}$, and also the highest coefficients of $\Omega_n (\lambda)$, $\tilde{\Omega}_n (\lambda)$ are self conjugate and regular:
\begin{align}
\langle \Omega_k | R \rangle &= \delta_{nk} \cdot I, \quad n, k = 0, 1, 2, \ldots
\end{align}
We introduce the auxiliary polynomials $\Phi_j (\lambda), \tilde{\Phi}_j (\lambda)$ of degree $j$ whose highest coefficients are equal to $I$, and which are such that
\begin{align}
\Omega_j (\lambda) &= \sum_{i=0}^{n} \tilde{C}_{j,i} \Phi_j (\lambda) D_{j,i}, \\
\tilde{\Omega}_j (\lambda) &= \tilde{D}_j (\lambda) D_{j,i},
\end{align}
where $C_{j,i}, D_{j,i}$ are the unknown highest coefficients of $\Omega_j (\lambda), \tilde{\Phi}_j (\lambda)$. It follows from conditions a) and b) that
\begin{align}
\langle \lambda k | R \rangle &= 0, \quad k < n, \\
\langle \Phi_k (\lambda) | R | \lambda k \rangle &= 0, \quad k < n.
\end{align}