ON THE EXTENSION OF A GENERALIZED DIFFERENTIATION OPERATOR

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A generalized differentiation operator introduced by A. F. Leont'ev is considered in this note. The concept of an LD-operator is introduced, and necessary and sufficient conditions are determined for the extension of a generalized differentiation operator to some LD-operator.

Let \( \{F_0(z)\} \) be some system of entire functions. We will consider the class of functions \( F(z) \) which are representable in the form

\[
F(z) = \sum_{n=0}^{\infty} d_n F_n(z)
\]

(each function in its own domain). A. F. Leont'ev (see [1]) introduced the operator

\[
D F(z) = \sum_{n=0}^{\infty} d_{n+1} F_n(z).
\]

We will call it a generalized differentiation operator (abbreviated g.d.o.). We note that if

\[
F_n(z) = z^n / n! \quad (n = 0, 1, ...),
\]

then \( DF(z) \) is the usual derivative of the function \( F(z) \). In the case

\[
F_n(z) = a_n z^n \quad (n = 0, 1, ...),
\]

where

\[
a_n \neq 0 \quad (n = 0, 1, ...), \quad \lim_{n \to \infty} n^{1/p} \sqrt[n]{|a_n|} = (\varepsilon p)^{1/p}, \quad \varepsilon > 0, p > 0,
\]

the operator \( DF(z) \) becomes the Gel'fond-Leont'ev operator (see [2]). In this case \( F(z) \) is an arbitrary function which is analytic in a neighborhood of the origin. In [3], A. F. Leont'ev showed that if

\[
F_n(z) = \frac{x^n}{(1 \cdot 2 \cdot ... \cdot n)} \quad (n = 1, 2, ...),
\]

then the function \( DF(z) \) is analytic in the entire domain of analyticity of \( F(z) \). In connection with this, A. F. Leont'ev posed the problem of determining all generalized differentiation operators which have this property. This problem was completely solved by Yu. F. Korobeinik [4] in the case \( F_n(z) = a_n z^n \). He introduced the notion of L-operator. An operator \( Py(z) \) is an L-operator if it satisfies the following conditions:

1) Let \( G \) be an arbitrary domain of the complex plane. Let \( A(G) \) denote the space of functions which are analytic and single-valued in the domain \( G \) with the topology of uniform convergence on compact sets. Then \( Py(z) \) is a continuous linear operator which acts from \( A(G) \) into itself for all \( G \).

2) If \( y(z) \) is analytic at the origin of the coordinate system, then for sufficiently small \( |z| \), \( Py(z) \) coincides with a g.d.o. \( Dy(z) \).


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If an L-operator $P_y(z)$ exists for a given g.d.o. $D_y(z)$, then we will say that the g.d.o. $D_y(z)$ can be extended to an L-operator. The result of Yu. F. Korobeinik can be stated as:

A g.d.o. $D_y(z)$ can be extended to an L-operator $P_y(z)$ if and only if the function

$$\omega(z) = \sum_{n=0}^{\infty} \frac{a_{n-1}}{a_n} z^{n-1} \quad (P_n(z) = a_n z^n)$$

is analytic in the domain $|z-1| > 0$ and has at least a second order zero at infinity. If this condition is satisfied, then the corresponding L-operator has the form

$$P_y(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{y(t)}{\partial z} \underbrace{\omega \left( \frac{z}{t} \right)}_{z \to 1} dt,$$

where $C_z$ is any rectifiable Jordan curve enclosing the point $z$ and lying together with its interior in the domain of analyticity of the function $y(z)$.

The present note concerns the solution of the same problem in the case when the $\mathcal{P}_n(z)$ are entire functions forming a basis in the disk $|z| < R$.

A. I. Markushevich proved the following theorem (see [5]): a system $\{\mathcal{P}_n(z)\}$ of functions analytic in $|z| < R$ is a basis in $|z| < R$ if and only if:

a) the system $\{\mathcal{P}_n(z)\}$ is a basis in the broad sense of the word;

b) there exists an expansion

$$\frac{1}{\xi - z} = \sum_{n=0}^{\infty} \omega_n(\xi) \mathcal{P}_n(z),$$

where the $\omega_n(\xi)$ ($n = 0, 1, \ldots$) are functions which are analytic for $|\xi| > R$, and where $\omega_n(\xi) = 0$ ($n = 0, 1, \ldots$). Moreover, the series converges uniformly for $|\xi| \geq R$, $|z| \leq r$ for any $r$, $0 < r < R$.

We apply the operator $D$ to the function $\frac{1}{z - \xi}$, $|\xi| > R$:

$$D \left( \frac{1}{z - \xi} \right) = \mathcal{F}(z, \xi) = \sum_{n=0}^{\infty} \omega_n(\xi) \mathcal{P}_n(z). \quad (1)$$

In the sequel, we will assume that the system $\{\mathcal{P}_n(z)\}$ in addition to the fact that it is a basis in $|z| < R$, satisfies the additional condition: the series (1) is uniformly convergent in $\xi$ in the region $|\xi| > R$ for sufficiently small $|z|$. It follows from this condition (see [6]) that if $\mathcal{F}(z)$ is an arbitrary function which is analytic in $|z| \leq R$, then the series $\sum_{n=0}^{\infty} d_n \mathcal{P}_n(z)$ determining $D \mathcal{F}(z)$ converges for sufficiently small $|z|$ and

$$D \mathcal{F}(z) = \frac{1}{2\pi i} \oint_C \mathcal{F}(t) \mathcal{F}(z, t) dt, \quad (1')$$

where $C$ is a closed rectifiable Jordan contour which lies together with its interior in the domain of analyticity of $\mathcal{F}(z)$ and contains the disk $|z| \leq R$ inside of it. In the case when $\{\mathcal{P}_n(z)\}$ is a quasi-exponential basis ([7]), the indicated property is automatically satisfied. Namely, it was proved in [6] that if the system $\{\mathcal{P}_n(z)\}$ forms a quasi-exponential basis in $|z| < R$, then the series (1) converges uniformly in $\xi$ and in $z$, respectively, in the region $|\xi| > R$, $|z| \leq r < R$.

§1. The L$D$-operator and its Properties. In the sequel, the expression "analytic function in the domain $G$" will mean a single-valued analytic function in this domain. If $y(z)$ is not a single-valued function, we will consider some branch of it in an appropriate domain.

Let $D$ be an arbitrary domain of the complex plane. The operator $P_y(z)$ is an L$D$-operator if it satisfies the following condition:

Let $A(G)$ denote the space of functions which are analytic in the domain $G$ with the topology of uniform convergence on compact sets. Then $P_y(z)$ is a continuous linear operator which acts from $A(G)$ into itself for all $G$ contained in $D$.

**Theorem 1.** Any L$D$-operator has the form

$$P_y(z) = \frac{1}{2\pi i} \oint_C y(t) \mathcal{F}(z, t) dt,$$