PARTITIONS AND CENTRALIZERS IN THE THEORY
OF FINITE GROUPS

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In the theory of groups, as of other algebraic structures, one of the central problems is the study of the mutual dependence between a group and its subgroups. On the one hand it is important to establish whether there exist in the given group subgroups of one or another form. On the other hand, given a system of subgroups we have to discover the structure of the group itself. Here the given systems of subgroups may be of the most varied kinds—from just a single subgroup to some kind of cover basis. Methods of prescribing a system of subgroups may be classified according to the properties of the subgroups in the system, depending on whether the internal properties of these subgroups are being considered or their properties in relation to the group itself or to other subgroups.

Our main attention in the dissertation will be devoted to the second part of the problem, and in particular to the study of the structure of a group in relation to a given system of centralizer elements. We are here considering, as a rule, finite groups. All the results transfer easily to locally finite groups by means of a well developed method already known. In the first chapters this is done in detail, but in the later ones the possibility is merely indicated, and the corresponding result is not even formulated.

A group G is said to be partitionable if there is in it a system \( \mathcal{R} \) of proper subgroups \( U_\alpha \) such that every nonidentity element of G is contained in one and only one subgroup \( U_\alpha \) of \( \mathcal{R} \). The system \( \mathcal{R} \) here is said to be a partition (or a basis of partition) of the group G, and the subgroups \( U_\alpha \) of the system \( \mathcal{R} \) are said to be components of the partition. If all the components of the partition are locally cyclic (Abelian, locally nilpotent, nilpotent), then the group is said to possess a complete (respectively, Abelian, locally nilpotent, nilpotent) partition. Groups with complete partition are also said to be completely partitioned. It is natural to relate locally cyclic groups to completely partitioned groups. Analogously, it is natural to assume that an Abelian (locally nilpotent, nilpotent) group possesses an Abelian (respectively, locally nilpotent, nilpotent) partition. Every partitionable group possesses what is known as an irreducible partition, i.e., a partition such that its components cannot be partitioned. It is convenient in speaking of partitions to restrict the term to the irreducible partition. An irreducible partition is unique and characteristic, i.e., an arbitrary automorphism of the group transfers a component of an irreducible partition to another component of an irreducible partition.

Frobenius groups are an important class of partitionable groups. In 1901 Frobenius [19] proved the following celebrated theorem:

Let a subgroup H of a finite group G be self-normalizing and be prime to each of the subgroups, distinct from H, which are conjugate with it. Then the set of elements which do not occur in any of the subgroups conjugate to H, together with the identity, is a subgroup F which is invariant in G.

In 1962 at the IVth All-Union Conference on General Algebra, V. M. Busarkin and A. I. Starostin [1] showed that this theorem transfers word for word to locally finite groups. Somewhat later this same fact was published by Kegel [20]. A locally finite group G which satisfies the conditions of this theorem is said to be a Frobenius group if H is a proper subgroup of G. F is said to be the invariant, and H (an arbitrary subgroup conjugate to H) is said to be complementary Frobenius factor. F and the subgroups conjugate to H constitute a partition of the group G. In the brief review of the theory of partitioned groups given in the first chapter of the dissertation, special attention is paid to Frobenius groups. In particular, a number of properties of Frobenius groups which are required in what follows are enumerated. Many generalizations of this theorem are now known. Some of them may be found in the review [7].

In 1906 Miller [22] indicated conditions for a finite Abelian group to be capable of partition, and in 1927 Young [28] established conditions for an arbitrary Abelian group to be partitionable. Weisner [27] considered groups in which the centralizer of each nonidentity element is Abelian. Such groups possess an Abelian partition.

The systematic study of partitions of groups was begun in the works of P. G. Kontorovich in the thirties and forties (see [5-7]). In these works he gives a general formulation of the problem of partitioning groups. The literature on partitionable groups is now very extensive.

Concrete examples of partitions are used in Chapter II of the dissertation in order to discover the characteristic subgroups of an arbitrary locally finite group—the kernels of a complete, an Abelian and a locally nilpotent partition. The kernel of a \( \Sigma \)-partition is the intersection of all maximal \( \Sigma \)-partitionable subgroups of the group, where \( \Sigma \) is one of the following group theory properties: complete partition, Abelian partition, locally nilpotent partition. It emerges that the kernel of a locally nilpotent partition of a locally finite group G is locally nilpotent if it is a proper subgroup of G, except in the case when G is a Frobenius group, the complementary factor of which is not locally nilpotent. In the exceptional case the kernel of the locally nilpotent partition is also a Frobenius group (Theorems II.1.2 and II.1.3).

The kernel of an Abelian partition of a locally finite group is Abelian if it is a proper subgroup of the group, except in the case when G is a Frobenius group, the invariant factor of which is Abelian or of prime exponent, and the complementary factor of which is noncyclic (Theorems II.2.4 and II.2.5). In the exceptional case, the kernel of the Abelian partition is also a Frobenius group and contains the invariant factor of the group itself.

A locally finite group G possesses a proper kernel (of a complete partition) \( K(G, z) \) which is not locally cyclic if and only if it is a group of one of the following types:

1. \( G = F \times H \) is a Frobenius group in which the invariant factor F is a noncyclic p-group of type p, and the complementary factor H is not locally cyclic. Here

\[
K(G, z) = F \cdot H_0,
\]

where \( H_0 \) is the center of the subgroup H.

2. \( G = (A \times \{b\}) \times \{c\} \), where \( A \times \{b\} \) is a Frobenius group with a locally cyclic invariant factor A, \( b^{pm} = c^m = e, m > 1. \)

Here

\[
K(G, z) = A \times \{b^{p^{m-1}}\}
\]

(Theorem II.3.5).

It is difficult to overestimate the part played in algebra by the concept of permutability. Considerable information on the structure of a group is provided by a knowledge of the centralizers of the elements. In the theory of finite unsolvable groups a special part is played by the centralizers of an involution. This is to some extent explained by the theorem of Feit and Thompson on the solvability of groups of odd order. A theoretically important fact about the finiteness of the set of finite simple groups with a given involution centralizer, established by R. Brauer [16], has defined an extensive program aimed at seeking abstract characterizations of well-known finite simple groups. Without such abstract characterizations it is difficult to imagine any further development of the theory of finite groups. In particular, they are also necessary, clearly, in order to solve the problem of describing all finite simple groups. In recent years many articles have appeared in this field. The present author does not propose to attempt a complete review of them.