EXTENSION OF QUASI-LIPSCHITZ SET FUNCTIONS

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In this article we consider so-called $\sigma$-triangular and quasi-Lipschitz set functions. In terms of $\sigma$-semimeasures, we establish necessary and sufficient conditions for extending a quasi-Lipschitz set function which is continuous from above at zero from a ring of sets to the $\sigma$-ring generated by these sets, and also conditions for the uniqueness of the extension. As simple corollaries we obtain analogous results for vector-valued measures, continuous triangular measures, and real-valued finite $\sigma$-triangular set functions which are continuous from above at zero.

Recently in the development of the theory of measures and abstract integrals much attention has been given to the study of nonadditive set functions (see [1-5] and [8]), which are related to the class of so-called triangular measures (see [4]) and triangular set functions (see [1]). The main interest of many authors has turned towards a classical aspect of the theory of set functions, namely, the question of extending nonadditive set functions by continuity (see [2, 3, 5, 8]). In joint research on questions concerning extensions of triangular set functions, Areshkin, Aleksyuk, and the author introduced the concept of an $\mathcal{N}$-triangular set function, Aleksyuk introduced a quasi-Lipschitz set function, and the author introduced a triangular measure (see Definitions 1 in Sec. 1, 1 in Sec. 2, and 2 in Sec. 2), the properties of which allow for a unified point of view in the study of conditions for extending vector, triangular, and outer measures.

In the present work, we use $\mathcal{N}$-semimeasures (theorem in Sec. 2) to establish a necessary and sufficient condition for the existence of an extension of a quasi-Lipschitz set function which is continuous from above at zero from a ring of sets to the $\sigma$-ring generated by these sets, and the extension is unique. The result which we obtain is valid for finite generalized measures, continuous triangular measures, and real-valued nonnegative finite $\mathcal{N}$-triangular set functions. Also, the theorem in Sec. 2 implies the fundamental result in [7], that is, Theorem 3 on necessary and sufficient conditions for extending vector-valued measures from a ring of sets to the $\sigma$-ring generated by these sets.

1. Basic Concepts and Auxiliary Assertions. Let $T$ be a set; let $M$ be a ring of subsets of $T$; let $S$ be the $\sigma$-ring generated by $M$. Let $(\mathcal{S}, |\cdot|)$ be a quasinormed abelian group; let $X$ be a Banach space; let $\mathbb{R}^+ = [0, +\infty)$; $\mathbb{R}^- = [0, +\infty]$.

A nondecreasing (nonincreasing) sequence of sets $\{E_n\}$ will be denoted by $E_n \nearrow (E_n \searrow)$, and we will write $E_n \nearrow E (E_n \searrow E)$ if and only if $E_n \nearrow (E_n \searrow)$ and $\lim E_n = E$. A sequence of sets $E_n \searrow \phi$ will be called a localizer, and a sequence of disjoint sets will be called a spectrum. A totally additive set function $\omega : M \to X$ will be called a vector-valued measure, that is, $\phi$ is a vector-valued measure if for each spectrum $\{E_k\} \subset M$ such that $\bigcup_{k=1}^{\infty} E_k = E \subseteq M$, we have $\varphi(E) = \sum_{k=1}^{\infty} \varphi(E_k)$.

Definition 1. A set function $\varphi : M \to (\mathcal{S}, |\cdot|)$, which equals zero on the empty set, is said to be $\mathcal{N}$-triangular if there exists a nonnegative real number $\mathcal{N}$ such that for arbitrary disjoint sets $A$, $B \in M$ we have...
It is easy to show that if $|\sigma| \neq 0$, then $\mathcal{M} \geq 1$. Thus we will always assume that $\mathcal{M} \geq 1$.

As examples of $\mathcal{M}$-triangular set functions with $\mathcal{M} = 1$ we have vector-valued and finite generalized scalar and outer (see [6]) measures. As an example of an $\mathcal{M}$-triangular set function with $\mathcal{M} = 2$ [for example, $(\mathcal{M} \neq 1)$] we have the set function $\phi$ defined by the condition

$$\phi(\{E\}) = \begin{cases} \mu(\{E\}) - 1, & \text{if } \mu(\{E\}) > 1, \\
\frac{1}{2\mu}(\{E\}) - 1, & \text{if } \mu(\{E\}) \leq 1,
\end{cases}$$

where $\mu$ is Lebesgue measure on $[0, 2]$ and $E \subset [0, 2]$.

Definition 2. An $\mathcal{H}$-valued or $(\mathcal{H}, |\cdot|)$-valued set function $\phi$ is said to be:

continuous from above at zero on $M$ if for an arbitrary localizer $E_n \uparrow \phi$ in $M$ we have

$$|\phi(E_n)| \to 0;$$

continuous from the side at zero on $M$ if for each spectrum $\{E_n\} \subset M$ we have (*)

continuous at zero on $M$ if (*) is satisfied for an arbitrary sequence of sets $\{E_n\} \subset M$ which converges to the empty set.

Definition 3. Following Aleksyuk, we will say that a family $\{\phi\}$, $\phi \in \mathcal{J}$, of $\mathcal{H}$-valued set functions satisfies the property of uniformly lacking escape loadings if for an arbitrary spectrum $\{E_k\} \subset M$ we have

$$\lim_{n \to \infty} \phi(E_k) = 0$$

uniformly with respect to $\alpha \in \mathcal{J}$. We will write $(\text{ULEL})_M$.

Definition 4. We will say that an $\mathcal{M}$-triangular set function $\phi: M \to (\mathcal{H}, |\cdot|)$ has the property $\mathcal{M}^{\mathcal{M}}$, if

We note that vector-valued measures, and also generalized and outer measures, have property $\mathcal{M}^{\mathcal{M}}$ (more precisely, property $\mathcal{M}^{\mathcal{M}_1}$).

Definition 5. By a supremization of a set function $\phi: M \to (\mathcal{H}, |\cdot|)$ we mean the set function $\tau$ defined by the condition

$$\tau(E) = \sup \{ |\phi(B)| : B \subset E, B \in M \}, E \in M.$$