THE FINAL $\sigma$-ALGEBRA OF AN INHOMOGENEOUS
MARKOV CHAIN WITH A FINITE NUMBER OF STATES

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It is proved that the final $\sigma$-algebra in the case of an inhomogeneous Markov chain with a finite number of states $n$ is generated by a finite number ($\leq n$) of atoms. The atoms are characterized from the point of view of the behavior of trajectories of the chain. Sufficient conditions are given (in the case of a countable number of states) that there should exist an unique atom at infinity.

Introduction. We consider an inhomogeneous Markov chain $x(t)$, $t = 0, 1, 2, \ldots$, with a finite number $n$ of states and transition functions $p_{ij}(s, t)$. Let $N_{t, \infty}$ be the minimal $\sigma$-algebra induced by the chain in time $[t, \infty)$, $N = \cap N_{t, \infty}$ the final $\sigma$-algebra, $P_{ti}$ measures on $N_{t, \infty}$ induced by the chain (i.e., conditional distributions under the condition $x(t) = i$). The set $B \in N_{t, \infty}$ is said to be null if $P_{ti}(B) = 0$ for all $t$ and $i$. The set $B \in N_{t, \infty}$ is said to be a nonnull (a.c.) atom if $B$ is nonnull and does not split into the sum of two nonnull nonintersecting sets. We know (see, e.g., [1]), that for a homogeneous Markov chain with a finite number of states, a trajectory of the chain, from some moment of time onwards, lies in one of a finite number of closed subsets of the states of the chain (an atom). For an inhomogeneous chain $x(t)$ a similar fact is true if we consider the trajectory at some nonrandom subsequence of moments of time. It is shown in this paper that any nonnull $N$-measurable set coincides almost certainly with respect to any measure $P_{ti}$, with the combination of a finite number ($\leq n$) of a.c. atoms. We can divide the space of states into subsets $E_j$ and find a sequence $\{t_k\}$, $t_k \to \infty$, such that each of the a.c. atoms coincides a.c. with one of the events:

$$\bigcup_{j} \bigcap_{k \geq t} x(t_k) \subseteq E_j.$$ 

We also give the sufficient conditions for "complete confusion at infinity," i.e., that the $\sigma$-algebra should consist of one a.c. atom. The sufficient conditions for "confusion" are given for the case of a countable number of states.

§ 1. THEOREM 1. The $\sigma$-algebra $N$ contains not more than $n$ pairwise nonintersecting nonnull sets.

Proof. Since the chain $x(t)$ is Markovian, for $A \in N$ we have

$$P_{ti}(A/N_{t, \infty}) = P_{ti}(A), \text{ a.c. } P_{ti},$$

(1)

where $N_{s,t}$ is the $\sigma$-algebra induced by the chain in time $[s, t]$. If we pass to the limit in (1) as $t \to \infty$, and use the convergence of the martingales [2] we obtain

$$\lim_{t \to \infty} P_{ti}(A) = P_{si}(A/N_{t, \infty}) = \chi(A), \text{ a.c. } P_{si},$$

(2)

where $\chi(A)$ is the characteristic function of the set $A$. If $A$ is nonnull, it follows from (2) and the finiteness of the number of states of the chain that there is a state $j$ and a subsequence $\{t_k\}$ of the sequence $\{t_i\}$, $t_i = 0, 1, 2, \ldots$, such that

$$\lim_{t_k \to \infty} P_{t_k}(A) = 1.$$
Consider a linear span over the set of vector functions of the form

\[ P_t(A) = \{P_{t1}(A), P_{t2}(A), \ldots, P_{tn}(A)\}, \quad A \in N. \]

We can show that the pairwise nonintersecting sets \( A_1, A_2, \ldots, A_l \) correspond with linearly independent vector functions

\[ P_i(A_1), P_i(A_2), \ldots, P_i(A_l). \]

It was shown above that there is a state \( j \) and a sequence \( \{t_k\} \) such that

\[ \lim_{k \to \infty} P_{t_k,j}(A_1) = 1, \]

Since \( A_2, A_3, \ldots, A_l \) are nonintersecting with \( A_1 \),

\[ \lim_{k \to \infty} P_{t_k,i}(A_2) = 0 \quad \text{for} \quad 2 \leq r \leq l. \]

Hence in the equality

\[ c_1P_1(A_1) + c_2P_2(A_2) + \ldots + c_lP_l(A_l) = 0, \]

\( c_1 = 0 \) and we can similarly verify that \( c_2 = 0, \ldots, c_l = 0 \). To prove the theorem it remains to show that there are not more than \( n \) linearly independent vector functions. Suppose for fixed \( s \) the basis for a linear span of vectors of the form

\[ P_s(A) = \{P_{s1}(A), P_{s2}(A), \ldots, P_{sn}(A)\}, \quad A \in N, \]

contains \( k(s) \) \((k(s) \leq n)\) vectors. From the Markovian property

\[ P_{si}(A) = \sum_{j=1}^{n} p_{ij}(s, t) P_j(A) \quad (4) \]

it follows that \( k(s) \) does not decrease as \( s \to \infty \). Since \( k(s) \not\equiv n \), then, beginning with some \( s_0 \), \( k(s) \) remains constant, \( k(s) = k \) for \( s \geq s_0 \). If \( \{P_{s_0}(A_1), A = 1, 2, \ldots, k\} \) is a basis for the above vector space at time \( s_0 \), \( \{P_{t}(A_1), A = 1, 2, \ldots, k\} \) remains a basis for any \( t \geq s_0 \), by (4). If we use the Markovian property (4) again, we find that in the expansion

\[ P_t(A) = c_1^tP_1(A_1) + c_2^tP_2(A_2) + \ldots + c_k^tP_k(A_k), \quad A \in N, \]

the constants \( c_1^t, c_2^t, \ldots, c_k^t \) can be assumed to be independent of \( t \). This shows that there are not more than \( k(k \leq n) \) linearly independent vector functions and the theorem is thus proved.

The assertion of Theorem 1 is similar to that of Lemma 2 [3], where an inhomogeneous Markov process, continuous to the right, with a finite number of states was discussed and the \( \sigma \)-algebra induced by the process to the left of the limit point of the discontinuities was studied.

It follows from Theorem 1 that there are a.c. atoms

\[ A_1, A_2, \ldots, A_k \quad (k \leq n) \]

such that their union forms a space of elementary events and any \( N \)-measurable set is either null or differs by zero from the combination of a number of the atoms \( A_1, A_2, \ldots, A_k \). Since the number of states is finite, by (3), for each a.c. atom \( A \) we can find a subset \( I = I(A) \) of the spaces of states and a subsequence \( \{s_m\} \) such that

\[ \lim_{m \to \infty} P_{s_m,i}(A) = 1, \quad i \in I, \quad (5) \]

but \( P_{s_m,i}(A) < 1 \) if \( i \not\in I \). Obviously, in (2) and (3) we can replace the sequence \( \{t'\} \) by any subsequence \( \{t''\} \), assuming now that \( \{t_k\} \) is a subsequence of \( \{t''\} \). Then for a finite number of a.c. atoms \( A_1, A_2, \ldots, A_k \) we can choose a single sequence \( \{s_m\} \) and a single number \( \alpha \) in (5). Obviously,

\[ I(A_1) \cap I(A_2) = \phi. \]

**THEOREM 2.** There is a sequence \( \{s_m\} \) and a partition of the space of states

\[ E = E_1 + E_2 + \ldots + E_j + \ldots + E_k \]

such that each a.c. atom \( A_j, 1 \leq j \leq k, \) almost certainly with respect to any measure \( P_{Si} \) coincides with the set

\[ \bigcup_{m \geq s_i} I(A_j) \subseteq E_j. \]