ASYMPTOTIC BEHAVIOR OF THE SPECTRUM OF CERTAIN PROBLEMS, CONNECTED WITH THE OSCILLATION OF FLUIDS

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One obtains the principal term of the asymptotics of the spectrum in a series of problems of the theory of small oscillations of fluids, filling partially a container. First one discusses a certain problem of general character, the "nonlocal Steklov type problem." The considered problems are connected with the oscillations of a capillary ideal (or capillary stratified) fluid, of a heavy viscous fluid, of a capillary viscous fluid, and with the oscillations of a fluid in container with an elastic bottom.

In [1] (see also [2, 3]), Kopachevskii has formulated a series of problems on the spectrum of the normal oscillations for certain physical models of fluids and, by heuristic considerations, has indicated asymptotic formulas for the spectrum in the formulated problems. In the present paper we justify the asymptotic formulas for the spectrum in all the problems from [1], which so far have not been solved. First we discuss a certain problem of a general character. A detailed exposition of most of the given results can be found in [4]. In [1], the formulations have been given in the form of boundary value problems on the eigenvalues and in the form of variational problems on the spectrum of a quotient of forms. Two circumstances are characteristic: 1) the presence of an "elliptic" constraint, 2) the nonsmoothness of the domain. For the justification of the asymptotic formulas we apply the method of the investigation of nonsmooth variational problems, developed by Birman and Solomyak in [5], and the technique of the pseudodifferential operators, allowing to study variational problems with elliptic constraints in the smooth situation (see [6]).

1. "Nonlocal" Steklov Type Problem. By $H^S$ one denotes the Sobolev spaces; $\mathcal{K}$ is a class of domains, satisfying the conditions of the usual imbedding and extension theorems (for $\Omega \in \mathcal{K}$ the Lipschitz property of $\partial \Omega$ is sufficient). Let $\Omega \subset \mathbb{R}^{m+1}$ be a bounded domain. We assume that the following condition is satisfied.

Condition 1. $\Omega \in \mathcal{K}$; the boundary $\partial \Omega$ contains a smooth $m$-dimensional surface $\Gamma$ with a smooth $(m-1)$-dimensional boundary $\gamma$.

By $\mathbf{n}(x)$ we denote the unit vector of the interior normal to $\partial \Omega$ at the point $x \in \Gamma$. Assume that in $\Omega$ there is defined the form $A[u]$ with smooth real coefficients

$$A[u] = \sum_{i,j=1}^{m+1} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + c(x)|u|^2 \ dx.$$  \hspace{1cm} (1)

The matrix $a(x) = \{a_{ij}(x)\}$ is uniformly positive in $\overline{\rho}, \rho(x) > 0$. Assume further that $q > 0$ is an integer, $B_T$ is a scalar, strongly elliptic differential expression on $\Gamma$ of order $2q$, $T_1, \ldots, T_q$ are differential trace operators, acting "from $\Gamma$ onto $\gamma,$" and $T_j \leq 2q - 1$. The coefficients of the operators $B_T$ and $T_j$, $1 \leq j \leq q$, are infinitely smooth. We assume that for $B_T, T_1, \ldots, T_q$ the Shapiro–Lopatinskii condition holds. By $B_T$ we denote the operator in $L^2(\Gamma)$, defined by the expression $B_T$ in the domain of definition $D(B_T) = \{u \in H^{2q}(\Gamma) : T_j u = 0 \text{ on } \gamma \}, j = 1, \ldots, q$. We assume that $\mathcal{B}_T$ is self-adjoint and positive definite. Then the compact operator $\mathcal{B}_T^{-1}$ is defined on $L^2(\Gamma)$. Let $\mathcal{B}(x, \xi), x \in \xi$, be the principal symbol of the expression $B_T$.

$$M(x, \xi) \overset{\text{def}}{=} \left[ (a(x) \xi \cdot \xi) (a(x) n(x) \cdot n(x)) - (\xi(x) \mathbb{I} n(x)) \right]^{1/2}$$

We consider the quotient of forms

$$\pm \frac{\mathcal{B}(x, \xi) u(u)}{\mathcal{A}(u)}, \quad u \in H^{1}(\Omega),$$

where the real-valued function $\mathcal{B} \in C^1(\Omega)$, while the number $r$ satisfies the conditions: $r > 1$ for $m = 1, r = \lambda+1/2$ for $1 < m < 2q + 1, r = \lambda + 1/2 + 1/2m$ for $m = 2q + 1, r = m/(2q + 1)^r$ for $m > 2q + 1$. We note that the extremals of the quotient (2) satisfy automatically the homogeneous elliptic equation $Lu = 0$ (in $\Omega$), where the operator $L$ corresponds to the form $A$.

By $\lambda^r_m (2)$ we denote the successive maxima of the quotient (2), while by $\mathcal{N}_\pm (\lambda(2)) = \text{card} \{ \lambda : \mathcal{N}_\pm (2) > \lambda \}$ we denote the distribution functions of the spectra. Below we shall use similar notations also for other variational quotients. If there is no negative spectrum, then we write simply $\mathcal{N}(\lambda(2)) = \mathcal{N}_+ (\lambda(2))$. Theorem 1. We have the asymptotic formula

$$\mathcal{N}_\pm (\lambda(2)) \sim \frac{2\pi}{m} \lambda^\theta \int dS(x) \int_{\xi(x) = 1} d\xi R_\pm (x, \xi),$$

where $\theta = m/(2q + 1)^r$, $R_\pm (x, \xi) \overset{\text{def}}{=} (\delta_\pm (x) B_\pm (x, \xi) M^\prime (x, \xi))^{1/2}, 2 \delta_\pm (x) \overset{\text{def}}{=} \mathcal{B}(x) \pm |\mathcal{B}(x)|$.

2. Application of Theorem 1 to Problems of Fluid Oscillations. Assume that $\Omega \subset \mathbb{R}^3$ satisfies Condition 1. Let $P$ be the orthoprojection in $L_2(\Gamma)$ onto $\hat{L}_2(\Gamma) = L_2(\Gamma) \Theta \{1\}$; $\Delta_\gamma$ is the Laplace–Beltrami operator on $\Gamma$, $\sigma > 0$ is a constant; $\mathbb{C} \in C^\infty(\Omega), \mathbb{C} \in C^\infty(\gamma)$ are real-valued functions. By $\mathcal{B}_T$ we denote the self-adjoint operator in $L^2(\Gamma)$, defined by the expression $P (\xi - \sigma \Delta_\gamma + a(x))$ in the domain of definition $D(\mathcal{B}_T) = \{u \in H^2(\Gamma) : \sigma \Delta_\gamma + a(x) u = 0 \text{ on } \gamma, \int_\Gamma u dS = 0 \}$. Here $\partial / \partial \nu\nu$ is the derivative along the normal to $\gamma$. We assume that $\mathcal{B}_T$ is positive definite. Then the operator $\mathcal{B}_T^{-1}$ is compact in $L^2(\Gamma)$.

The problem of the spectrum of the normal oscillations of a capillary perfect (or capillary stratified) fluid reduces (see [1, 2]) to the problem of the spectrum of the variational quotient

$$\frac{\int_\Gamma (\mathcal{B}_T^{-1} u) \overline{u} dS}{\int_\Gamma \mathcal{B}^{-1} (x)|u|^2 dx}, \quad u \in H^1(\Omega), \int_\Gamma u dS = 0.$$ (4)

Here $\mathfrak{q} \in C^\infty(\Omega), \mathfrak{q}(x) > 0$; for a perfect fluid we have $\mathfrak{q}(x) = \text{Const.}$

Proposition 1. We have the asymptotic behavior

$$\mathcal{N}(\lambda(\mathfrak{q})) \sim \lambda^{-2/3} \mathfrak{q}^{2/3} \int_\Gamma \frac{1}{\sigma} \mathfrak{q}^{2/3}(x) dS(x).$$ (5)