SOLVABILITY "IN THE SMALL" OF A NONSTATIONARY FLOW PROBLEM FOR AN IDEAL INCOMPRESSIBLE FLUID, I

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For the system of Euler equations and the incompressibility equation one considers the following initial-boundary value problem: the field of velocities $\mathbf{v}$ is prescribed at the initial moment and for all $t>0$ one gives the following boundary conditions: on the entire boundary of $\Omega$ the normal component of the velocity $\mathbf{v}$ is prescribed and on that part $\mathbf{s}_i$ of the boundary of $\Omega$ where inflow occurs one prescribes the value of the velocity $\mathbf{v}=\mathbf{v}|_{\mathbf{s}_i}$, whose components satisfy a certain necessary equality, derived in the paper. For such a problem one proves its unique solvability on a small interval of time.

The initial-boundary value problem for Euler's equation in a bounded domain with no flow through the boundary, has been investigated by many authors: for the first time in the thirties by N. M. Goryunov [1, 2], L. Lichtenstein [3], W. Wolibner [4] and presently in [5-13]. The initial-boundary value problem with flow through the boundary of the domain has been considered only for the two-dimensional case by V. I. Yudovich [14]. The flow problem in the three-dimensional case has been considered in [15]. This paper was published by Doligidze on the basis of the notes of N. E. Kochin. The results presented in it are incorrect since the problem itself which is posed in it is overdetermined and cannot be solved without additional restrictions on the functions given in the problem. In [15], on the part $\mathbf{s}_i$ of the boundary of the domain $\mathcal{R}$ of the flow where inflow occurs, one has prescribed an arbitrary vorticity vector $(\omega=\text{rot}\mathbf{v})$. We consider a similar problem prescribing on $\mathbf{s}_i$ the vector $\gamma=\omega|_{\mathbf{s}_i}$ tangent to $\mathbf{s}_i$, where the tangential components are subjected to the necessary constraint (1.18), derived below. For this problem we prove the unique solvability in a small time interval. The method of proof does not depend on the dimension of the physical space in which the fluid flows. In its main part, the proof is based on O. A. Ladyzhenskaya's paper [5].

The present paper has been completed during the apprenticeship of the author in the laboratories of Prof. O. A. Ladyzhenskaya at the Leningrad Branch of the V. A. Steklov Institute of Mathematics of the Academy of Sciences of the USSR. The author is grateful to Prof. O. A. Ladyzhenskaya for the formulation of the problem and for indicating the course of its investigation and to Prof. V. A. Solonnikov for useful discussions.

1. Formulation of the Problem

Let $\mathcal{R}$ be a bounded domain in $\mathbb{R}^3$ with boundary $\partial \mathcal{R}$ of class $C^3$. We consider the Euler equation in $\mathcal{R}_T=\mathcal{R}\times[0,T]$.

*In the paper we use the convention regarding summation with respect to repeated indices.

with the initial conditions
\[ \psi|_{t=0} = a(x), \quad \text{div} \psi = 0 \] (1.3)

and with boundary conditions of the form
\[ \mathbf{v}_n|_{S_1} = \mathbf{e}_n = \mathbf{e}_n \times 0, \quad \mathbf{v}_n|_{S_2} = \mathbf{e}_n \times 0, \quad \mathbf{v}_n|_{S_0} = 0, \quad \int \mathbf{e}_n \, ds = 0, \] (1.4)

where \( \partial \Omega = S_1 \cup S_2 \cup S_0 \), \( \mathbf{v}_n = \mathbf{v} \mathbf{n} \), \( \mathbf{n} \) being the vector of the exterior normal. In addition, we assume that the following consistency conditions hold:
\[ \mathbf{e}_n|_{S_1} = 0, \quad \mathbf{e}_n|_{S_2} = 0, \quad \mathbf{n} \cdot a(x)|_{x \in \partial \Omega} = \mathbf{e}_n(x,t)|_{t=0}. \] (1.5)

On the part \( S_1 \) of the boundary we have an inflow, while on \( S_2 \) we have an outflow from the domain \( \Omega \) and \( S_1 \cap S_2 = \emptyset \).

The boundary conditions (1.4), (1.5) are not yet sufficient for the correctness of the formulation of the problem (see [14, 15]). As an additional boundary condition we prescribe the vorticity vector on \( S_1 \). However, as it will be proved below, the curl vector on \( S_1 \) cannot be given arbitrarily; it must satisfy specific conditions.

Applying the operator \( \text{rot} \) to (1.1) and to the initial condition (1.3) and appending the missing boundary condition mentioned above, we obtain the problem:
\[ \omega_t + \mathbf{v} \times \omega = \mathbf{F} = \text{rot} \psi \] (1.6)
\[ \omega|_{t=0} = \omega_0(x) = \text{rot} a \] (1.7)
\[ \omega|_{S_1} = \eta(x',t) \quad x' \in S_1, \] (1.8)

where \( \omega = \text{rot} \mathbf{v} \) is the curl vector; condition (1.8) is the additional condition. The vector \( \mathbf{v} \) can be reestablished in terms of the curl \( \omega \) as the solution of the following overdetermined elliptic problem
\[ \begin{align*}
A. & \quad \text{rot} \psi = \omega \\
& \quad \text{div} \psi = 0 \\
& \quad \mathbf{v}_n|_{\partial \Omega} = \mathbf{t}.
\end{align*} \]

In order to solve the problem \( B \) for a given field \( \psi(x,t) \), we introduce the characteristic curves defined by the equations