STABILITY OF UNSTEADY VISCOUS FLOWS

Yu. M. Shtemler

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A study is made of the linear stability of plane-parallel unsteady flows of a viscous incompressible fluid: in the mixing layer of two flows, in a jet with constant flow rate, and near a wall suddenly set in motion [1]. The slow variation of these flows in time compared with the rate of change of the perturbations makes it possible to use the method of two-scale expansions [2]. The stability of nonparallel flows with allowance for their slow variation with respect to the longitudinal coordinate was investigated, for example, in [3-6]. The unsteady flows considered in the present paper have a number of characteristic properties of nonparallel flows [1], but in contrast to them are described by exact solutions of the Navier–Stokes equations. In addition, for unsteady plane-parallel flows a criterion of neutral stability can be uniquely established by means of the energy balance equation.

1. We consider the stability of the plane-parallel unsteady flow

\[ U(Y, T) = U_0 + Q(T) u(Y/\Lambda(T)), \quad V = 0, \quad P = P_+, \quad \Lambda(T) = (vT)^{1/2}, \quad Q(T) = \Gamma(vT)^{-b/2} \]  

(1.1)

Here, \( P \) is the pressure, \( U \) and \( V \) are the projections of the velocity vector onto the \( X \) and \( Y \) axes, \( T \) is the time, \( P_+ \) is the value of the pressure in the limit \( |Y| \rightarrow \infty \), \( \nu \) is the kinematic coefficient of viscosity, \( U_0 \) and \( \Gamma \) are dimensional constants and \( b \) is a dimensionless constant, all three depending on the specific flow.

For flows in a mixing layer, in a jet with constant flow rate, and near a flat wall suddenly set in motion, we have, respectively,

\[ U_0 = \frac{U_+ + U_-}{2}, \quad \Gamma' = \frac{U_+ - U_-}{2}, \quad b = 0, \quad u(y) = \text{erf} \left( \frac{y}{2} \right) \]  

(1.2)

\[ U_0 = 0, \quad \Gamma = \frac{f}{2\nu r^2}, \quad b = 1, \quad u(y) = \exp \left( -\frac{y^2}{4} \right), \quad \Gamma = -W, \quad b = 0, \quad u(y) = \text{erf} \left( \frac{y}{2} \right) \]

In the relations (1.2), \( U_+ \) and \( U_- \) are the values of the velocity as \( |Y| \rightarrow \infty \), \( W \) is the velocity of the wall, and \( \rho \) is the density of the fluid.

Suppose that at the time \( T = T_0 \) we superimpose on the main flow described by Eqs. (1.1) and (1.2) a perturbation whose flow function \( \psi'(X, Y, T) \) depends periodically on the longitudinal coordinate \( X \) with the dimensional period \( 2\pi/\kappa \). In view of a periodicity of the perturbations with respect to the longitudinal coordinate and the self-similarity of the undisturbed flow, it is convenient to go over to the dimensionless variables.
The linearized equation for the flow function of the perturbations, written down in the dimensionless variables (1.3), has the form

\[ \begin{align*}
\left( N - \frac{\partial}{\partial t} M \right) \psi' &= \varepsilon(t) M \psi', \\
M &= \lambda^*(t) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) S = -\frac{1}{2} \left( b + 1 + \frac{\partial^2}{\partial y^2} \right) M
\end{align*} \]

(1.4)

Here, \( \lambda^*(t) = \frac{\lambda(T)}{\Lambda(T)} \) and \( \lambda(T) = \Delta(T) \lambda^*(T) \) of the undisturbed flow.

The no-slip condition on the wall and the condition of damping in the exterior flow have the form

\[ \psi'(0, y) = 0 \quad \psi'(y, \pi) = 0 \quad (|y| \to \infty) \]

(1.5)

We introduce an auxiliary function \( \tau(t) \) of the time defined by

\[ \frac{d\tau}{dt} = \varepsilon(t), \quad \tau(0) = 0 \]

(1.6)

Using Eqs. (1.1) and (1.3), we readily show that

\[ \begin{align*}
\lambda^*(t) &= \lambda(t), \\
q^*(t) &= q(t), \\
\varepsilon^*(t) &= \varepsilon(t)
\end{align*} \]

(1.7)

Note that in Eq. (1.6) \( \varepsilon^*(t) = \varepsilon(t) \ll 1/\lambda^0 \).

Below, we shall assume that the Reynolds number \( \lambda^0 \) is fairly large, so that the velocity of the main flow varies slowly compared with the velocity of the perturbed flow; then \( \lambda, q, \lambda^0 \), and \( \varepsilon \) are slow functions of the time.

2. To investigate the stability of these flows, we use the method of two-scale expansions [2]. We seek a solution to the problem (1.4)-(1.5) which is periodic with respect to the longitudinal coordinate in the form

\[ \psi'(x, y, t) = \exp(\omega(t)) \psi(x, \tau) \exp(\theta(\tau)) \]

(2.1)

Here, the phase \( \theta \) is the "fast" variable and \( \tau \) the "slow" variable, and the normalizing factor \( \omega/\tau \) is introduced in order to make \( \psi(y, \tau) \) the amplitude function of the transverse component of the velocity; \( \omega(t) = \kappa(\lambda(t), \lambda^* = \kappa^* \lambda(\tau)) \). The connection between the variables \( x, t \) and \( \theta, \tau \) is given by

\[ \begin{align*}
\lambda'(t) &= \kappa(\lambda(t), \lambda^* = \kappa^* \lambda(\tau)), \\
\kappa(\lambda(t)) &= \kappa(\lambda(t)) \phi(\tau), \\
\frac{d\theta}{dt} &= \varepsilon(t), \quad \tau(0) = 0
\end{align*} \]

(2.2)

In Eqs. (2.2), \( \omega(\tau) \) is the unknown proper time.

Substituting (2.1) and (2.2) in Eq. (1.4), we obtain

\[ \begin{align*}
(N_1 - i\omega M_1) \psi &= \varepsilon \left( G_1 + M_1 - \frac{\partial}{\partial \tau} \right) \psi, \\
G_1 &= S_1 - \frac{1}{2} \left( \frac{\partial^2}{\partial y^2} + \kappa^2(\tau) \right)
\end{align*} \]

(2.3)

Here, the operators \( N_1, M_1, \) and \( S_1 \) are obtained from \( N, M, \) and \( S \) by replacing \( \lambda^*(t) \partial/\partial x \) by \( i\omega(\tau) \).

We shall seek \( \omega(\tau) \) and \( \psi(y, \tau) \) in the form of series in \( \varepsilon(\tau) \):

\[ \begin{align*}
\omega(\tau) &= \omega_0(\tau) + \varepsilon(\tau) \omega_1(\tau) + \ldots, \\
\psi(y, \tau) &= \psi_0(y, \tau) + \varepsilon(\tau) \psi_1(y, \tau) + \ldots \]
\]

(2.4)