DIFFERENTIAL AND DIFFERENCE ALGEBRA

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In a survey the main results in differential and difference algebra (differential rings and modules; differential fields; differential-algebraic geometry; differential and difference dimension; differential and difference Galois theory; integration in finite terms; differential algebraic groups), compiled from the reviews published in Referativnyl Zhurnal Matematika for the years 1975-1986, are reflected.

This survey has been compiled primarily from the reviews published in Referativnyl Zhurnal Matematika in 1975-1986. In a certain sense, Kolchin's monograph [282] is the starting point for the survey (it contains an extensive bibliography up to 1973, which is not duplicated here, unless necessary for completeness of presentation).

Space limitations have forced us to leave out a number of topics which are clearly related to differential and difference algebra. Thus, the work on the theory of differential and difference rings is only partially covered. These topics have been treated by a number of surveys in the theory of rings and modules (see, e.g., [4] and [5]). A large cycle of publications on the structure of the ring of differential operators and its homological properties have been specifically excluded (this topic is dealt with in the monograph of Bjork [116]). The theory of p-adic differential equations, which has been actively developing during the recent years in the framework of nonarchimedean analysis (see [191]), also remains outside the scope of this survey. Finally, the problem of construction and analysis of symmetries of (partial) differential equations and questions of integrability of these equations are relevant to many researchers in the fields of differential geometry, theory of differential equations, computer algebra, and theoretical physics. The literature on this question is voluminous and it is virtually ignored in this survey (see, e.g., the survey of Vinogradov [21]).

There are several different points of view on the subject of differential algebra. Kaplansky's monograph [42] begins with the following sentence: "Differential algebra is easily described: it is (99% or more) the work of Ritt and Kolchin." Vasil'ev [19] defines differential algebra as a division of differential geometry studying the symmetries of differential equations. Kolchin [282] maintains that it is very difficult to separate clearly differential algebra from the related areas in algebra and differential equations and "in principle, every work on algebraic differential equations belongs to differential algebra" [282, p. XV]. The dividing line depends on the personal preferences of the author. It is no less difficult to trace the dividing line between differential algebra and differential geometry, algebraic geometry, computer algebra, homological algebra, and other areas of mathematics. In view of all this, the selection of material in this survey is largely subjective. Everything that we have said about differential algebra is equally true for difference algebra.

The first monographs on differential algebra are the books of Ritt [410-411]. Here differential algebra was first defined as a separate branch of mathematics. Ritt also posed a large number of problems, many of which remain unsolved to this day. Kolchin, in his paper at the Moscow Congress of Mathematicians in 1966 [281], defined several directions of differential algebra and stated some open problems which largely shaped the development of differential algebra in the last 20 years. The monograph [282] was a state-of-the-art survey of most of these topics at the time of its publication. It has preserved its relevance to this day.

There is a paucity of monographs and textbooks on differential algebra. In addition to the books of Ritt, Kaplansky, and Kolchin cited above, we should mention the recent monograph Translated from Itogi Nauki i Tekhniki, Seriya Algebra, Topologiya, Geometriya, Vol. 25, pp. 67-139, 1987.
of Kolchin [287] on differential algebraic groups, the monograph by Matsuda [329], and the book of Pommaret [382] on differential Galois theory, although Pommaret himself holds the opinion that differential Galois theory is part of differential geometry, and not differential algebra. Most of the results on differential algebra have been published only in journal articles.

The situation in difference algebra is largely the same. It also has its origin in the papers of Ritt, followed by the monograph of R. Cohn [165]. The development of difference algebra generally follows the development of differential algebra: some time after the publication of each result in differential algebra, the analogous result in difference algebra is published. This was the case with the basis theorem, the Galois theory, the theory of integration in finite terms (its difference analogue is the theory of summation in finite terms), and so on. Note that the analogy is far from complete: all these results in difference algebra are essentially different from the corresponding results in differential algebra. This survey will focus on differential algebra, and the presentation of each group of results for the differential case will be followed by the corresponding results for the difference and differential-difference cases.

In addition to lack of clear boundaries, differential-difference algebra suffers from lack of uniform terminology. The same results may be presented in different "languages," as noted by Manin [57]. On the other hand, different authors use the same terms in entirely different meanings, e.g., the basic notion of differential polynomial ring usually stands for a commutative ring generated by a finite set of indeterminates and the denumerable set of all their derivatives. Yet some authors use the term "differential polynomial ring" for the ring of linear differential operators. In this survey, we mainly follow the terminology of Kolchin's books.

1. DIFFERENTIAL RINGS AND MODULES

The objective of this section is to provide a general view of the main directions of research on differential rings and modules. Supplementary material can be found in specialized surveys of the theory of rings and modules (see, e.g., [4, 5]).

1.1. Description of Derivations of Rings. Jacobson [237] showed that each derivation of a finite-dimensional simple algebra over a field is inner if either the field is of characteristic zero or the center of the algebra coincides with the base field. The case of positive characteristic was considered by Baer [99]. The derivations of primitive algebras with nonzero socle were described in [38]. Kaplansky [257] proved that the dimension of the derivation algebra Der A of the n-dimensional algebra A with nonzero multiplication is at most $n^2 - n$ (the bound $n^2 - n$ is attained on some classes of algebras). The derivations of matrix rings over nonassociative algebras were dealt with by Benkart [103] and Benkart and Osborn [104]. The derivations of quasimatrix rings, incidence rings, and their generalizations were classified by Murase [346], Baclawski [98], Burkov [9], and Abdeljaonad [91]. Nowicki [357, 358] considered the derivations of matrix subrings. Okugawa [363] showed that the polynomial ring over a ring of positive characteristic has infinitely many pairwise commuting derivations (this is not true for a ring of characteristic zero). Burkov [13-15] described the derivations of the skew polynomial ring. Sprengelmeier [473] analyzed the derivations and automorphisms of finite-dimensional algebras in which $x^2 = 0$ implies $x = 0$. Smith [471] described the derivations of group algebras of finitely generated nilpotent groups without torsion. Burkov [10, 16] described the derivations of group algebras of periodic groups. Rota, Sagan, and Stein [425] and Reutenauer [401] studied cyclic derivations of the noncommutative algebra of formal power series. The conditions when the additive mapping $D: R \to R$ such that

$$D(x^n) = \sum_{j=1}^{n} x^{n-j} D(x) x^{j-1} \quad (n \geq 2 \text{ is fixed})$$

is a derivation in a ring were obtained by Bridge and Bergen [127].

Posner [384] showed that the composition of two nonzero derivations of a prime ring of characteristic other than 2 is not a derivation. The properties of the composition of many derivations were studied by Krempa and Matczuk [299]. Burkov's work on the composition of derivations is closely related. Cusack [176] showed that all Jordan derivations are derivations.