AN EXACT SOLUTION OF TIME-DEPENDENT EQUATION OF
TRANSFER OF TRAPPED RADIATION IN A FINITE
ABSORBING MEDIUM

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Abstract. Milne's time-dependent equation of transfer of trapped radiation in a finite absorbing medium has been exactly solved by a combination of Das Gupta's modified form of the Wiener-Hopf technique and method of solving boundary value problems in the theory of heat conduction as adopted by Chandrasekhar (1950) to solve the same equation approximately in combination with his method of discrete ordinates. Milne himself had earlier obtained an approximate solution of his equation.

The problem of diffusion of imprisoned resonance radiation through a gas was first considered by Compton (1922) followed by Milne (1926) and Kenty (1932). Holstein (1947) treated the decay of resonance radiation in optically-excited gases for the case of Doppler-broadened radiation in plane-parallel enclosures. He attacked the problem in a more fruitful way through detailed elaboration of physical aspects, by considering the radiative transport of excitation and formulating a general transport problem in terms of a Boltzmann-type integro-differential equation involving $T(p)$ which defines the probability that a quantum traverses a layer of gas of thickness $p$ without being absorbed. This approach was adopted in view of the fact that, as Holstein (1947) discussed, it is not possible to define a mean-free path for resonance quanta and, hence, to describe the radiative transport of excitation by a diffusion equation as was done by the earlier investigators. A steady-state solution of the equation was then obtained by Holstein by the Ritz variational method and the rate of decay of excitation in an infinite slab is then evaluated and compared with the Zemainsky's measurement of the decay of radiation from an enclosure of mercury vapour which emits resonance lines 2537Å and 1849Å through transition from an excited state to the ground state (see Zemainsky, 1932).

Holstein continued the study of imprisoned resonance radiation in gases in another paper (see Holstein, 1951) for the case of a second type of enclosure geometry-infinite cylinders extending his treatment to a variety of spectral line shapes corresponding to (a) Doppler broadening, (b) pressure broadening, (c) impact broadening, and (d) statistical broadening.

In the decay experiments (studying imprisonment of resonance radiation) an optically excited gaseous system is irradiated for some time with an incident beam of resonance radiation and then abruptly cut-off from the source. After the incident beam has been off the intensity of the diffuse radiation is observed to undergo an exponential decay of

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the form $e^{-\nu t}$, where $1/\nu$ is the radiative lifetime of an atom and $1/g$ in a way gives the number of emissions and absorptions of an individual unit of atomic excitation prior to its escape from the gaseous enclosure. Holstein, in this paper, computes values of $g$ the ‘escape factor’ (as Holstein calls it) for the various line shapes (as mentioned above) and for different enclosure geometries. Kibble et al. (1967) investigated the effect of resonance trapping on the lifetimes of the $3^2P$ levels of sodium for a range of sodium vapour pressures from $10^{-7}$ to $10^{-4}$ Torr. Their results are then compared with the theories of Holstein (1947, 1951) and Milne (1926) by fitting the theoretical expressions to the data. Milne’s theory is seen to be in good agreement with experimental values over the whole pressure range, Holstein’s expressions on the other hand are in agreement only at pressures above $5 \times 10^{-5}$ Torr where the condition of optical depth $>> 1$ assumed by Holstein is satisfied.

The resonance lines are absorbed quite appreciably even by a very small layer (of the order of 0.001 cm) of gas in its normal state. Therefore, the probability of a resonance quantum emitted by an atom, being again absorbed by another atom within a very short distance of transit is very high and the atom so excited again emits the quantum. The final escape of a quantum across the boundary of the gaseous enclosure may, therefore, require a large number of such emission and reabsorption of the quantum by atoms leading to transfer of excitation energy. Radiation energy locked in such ways in the gas is said to be imprisoned.

However, it was Milne (1926) who considered the escape of imprisoned radiation as a problem in radiative transfer and derived the appropriate transfer equations calculating from the Einstein’s $A, B$ coefficients of absorption and emission; and these basic transfer equations Milne solved in the approximation of Schuster and Schwarzschild (see Chandrasekhar, 1950, art. 20). In the same source, Chandrasekhar (1950, art. 90) in treating this problem of imprisoned radiation banked on this basic transfer equation derived by Milne, introduced an approximation consistent with the practical situation that exists in a passive radiative gaseous medium to derive transfer equations of practical interest appropriate for the case under consideration. Chandrasekhar then proceeded to treat this equation as a boundary-value problem in a new manner similar to analogous boundary-value problems in the theory of heat conduction, and finally reduced the equation to an integro-differential equation which is then approximately solved by the method of discrete ordinates. The solution obtained by Chandrasekhar in the first approximation comes out to be equivalent to the approximate solution derived by Milne (1926).

In the present paper we give an exact solution of the transfer equation of Milne describing the diffusion of imprisoned radiation in a gas by a method based on the combination of the method applied to boundary value problems in heat conduction theory (akin to Chandrasekhar’s adaptation) and Das Gupta’s method based on Laplace transformation combined with Wiener–Hopf technique, borrowing results from the exact solution of the transport equation in finite multiplying media (see Das Gupta et al., 1980, 1981). We consider a plane-parallel slab of gas (say, of mercury vapour) of breadth $x_0$ being illuminated from the side at $x = 0$ by an incident radiation field from