ON THE FIRST BOUNDARY PROBLEM FOR A HYPERBOLIC EQUATION IN AN ARBITRARY CYLINDER

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A study is made of the solutions of a second-order hyperbolic equation which vanish on the boundary of an arbitrary domain in the space of the variables $x_1, \ldots, x_n$. The degree of smoothness in the initial conditions, necessary and sufficient to guarantee the same degree of smoothness in the solution (considered as a function of $x_1, \ldots, x_n$) for all $t$, is ascertained.

1. It is known (see [1], pp. 531-532) that a solution of the Cauchy problem for the wave equation

$$u_{tt} = \Delta u, \quad u|_{t=0} = \psi, \quad u_t|_{t=0} = \Psi$$

possesses the following property. If for some $t$ and for every finite domain $G'$ of the $x_1, \ldots, x_n$-space, we have $u \in W^2_k(G')$ and $u_t \in W^1_{k-1}(G')$, then the same will be true for every $t$. A solution of the first boundary problem for a domain with a non-smooth boundary in the case $n \geq 2$, $k \geq 2$, will, in general, not possess this property (see example 1).

Example 1. In the domain $G (0 < r < \infty, 0 < \varphi < 2\pi)$ with the boundary $\Gamma (0 \leq r < \infty, \varphi = 0$ and $\varphi = 2\pi)$, where $r, \varphi$ are polar coordinates, the function

$$u = r^{-1/2} \left[ f(t + r) - f(t - r) \right] \sin \frac{\varphi}{2}$$

for arbitrary $f \in C^2$ is continuous in $G + \Gamma$, satisfies the wave equation $u_{tt} = \Delta u$, and the boundary condition $u|_{\Gamma} = 0$; in every finite domain $G'$, $u \in W^1_k(G')$.

Let $f$ be an arbitrary $C^\infty$-function, which vanishes outside the interval $(1, 3)$, and for which $f' \neq 0$ in the intervals $(1, 2)$ and $(2, 3)$. Then for $t \leq 1$ and $t \geq 3$, the solution (2) belongs to $C^\infty$ and to $W^1_k(G')$; furthermore,

$$u_{tt} = -\frac{1}{2} r^{2} f''(t) + O(r^{2})$$

and, for $1 < t < 2$ and $2 < t < 3$, the solution does not belong to $W^1_k(G')$, $G' = G(r < 1)$.

In this note we shall determine classes of functions $D_k$ having the property that if for some $t$ a solution $u$ of the first boundary problem, considered as a function of $x_1, \ldots, x_n$, satisfies the relations $u \in D_k$, $u_t \in D^{k-1}$, then it satisfies these relations for all $t$.

We shall in this note employ the ideas introduced in [2], and in chapter 2 of [3].

2. Let $G$ be a domain, finite or infinite, in the space of the variables $x_1, \ldots, x_n$; let $\Gamma$ denote the boundary of $G$; let $x_\infty (x_1, \ldots, x_n)$. Let us denote by $D^0$ the class of functions, defined in $G$ and belonging to $L^2$ in each finite portion of domain $G$. Further, let $D^1$ be the closure, in the metric $W^2_k$, of the set of those functions of $C^1(\overline{G} + \Gamma)$, which vanish near $\Gamma$. Thus, $D^1$ coincides with the class $D$ given in [3], p. 38, and with the class $W^1_k$ given in [4], p. 20.

Let $k \geq 2$. We shall say that $u \in D_k$, if $u \in D^1$, and if $u$ has in $G$ a generalized derivative of the second order, and if, in addition, $\Delta u \in D^{k-2}$.

In accord with [4], pp. 219-220, this definition is equivalent to the following: $u \in D_k$ if $u$ is a generalized solution of the problem.

\[ \Delta u = -f \text{ in } G, \ u_{|_\Gamma} = 0 \quad (4) \]

for some \( f \in D^{k-2} \).

Here (as in [3], p. 197), by a generalized solution of problem (4) for an arbitrary \( f(x) \in D^0 \), we shall mean a function \( u(x) \in D^1 \), which satisfies the equation
\[
\int_G \nabla u \cdot \nabla \Phi \, dx = \int_G f \Phi \, dx
\]

for every function \( \Phi(x) \in D^1 \) (finite, if \( G \) is infinite). Here \( \nabla \) is the gradient operator, and \( \nabla u \cdot \nabla \Phi \) is a scalar product.

3. Let us assume now that domain \( G \) is finite. By a generalized characteristic function of the problem
\[
\Delta v = -\lambda v \text{ in } G, \ v_{|_\Gamma} = 0 \quad (6)
\]

we shall understand a function \( v \in D^1 \), which satisfies Eq. (5) for \( u = v, f = \lambda v \), for every \( \Phi \in D^1 \) (as in [3], p. 44).

It is known (see [3], p. 45) that there exists for problem (6) a complete orthonormal system of characteristic functions \( v_i, i = 1, 2, \ldots \); all \( \lambda_i > 0, \lambda_i \to \infty (i \to \infty) \),
\[
\int_G v_i v_k \, dx = \delta_{ik}, \quad \int_G \nabla v_i \cdot \nabla v_k = \lambda_i \delta_{ik} \quad (7)
\]

\( \delta_{ik} = 0 (k \neq i), \delta_{ii} = 1 \). We shall denote the Fourier coefficients of function \( f \) with respect to the system \( v_i \) by
\[
\gamma_i = \int_G f v_i \, dx \quad (8)
\]

As is known, statements a), b), c) which follow are equivalent: a) \( f \in L^2 \); b) \( \sum \lambda_i^2 < \infty \); c) \( \Sigma \gamma_i v_i \) converges in \( L^2 \). Moreover,
\[
\sum \gamma_i^2 = \int f^2 \, dx, \quad \sum \gamma_i v_i = f \quad (9)
\]

**Lemma 1.** The Fourier coefficients of a generalized solution of problem (4) are equal to \( c_1 = \gamma_i / \lambda_i \).

**Proof.** Let us take, in the definition of a generalized characteristic function \( v_i, \Phi = u \). We obtain
\[
\int_G \nabla u \cdot \nabla u \, dx = \lambda_i \int_G v_i u \, dx = \lambda_i c_i \quad (10)
\]

By virtue of Eq. (5) with \( \Phi = v_i \), and Eq. (8), the left side of Eq. (10) is equal to \( \gamma_i \).

**Lemma 2.** The following statements are equivalent: a) \( f \in D^1 \); b) \( \Sigma \lambda_i^2 < \infty \); c) \( \Sigma \gamma_i v_i \) converges in \( W^1_2 \). Moreover,
\[
\sum_{i=1}^\infty \lambda_i^2 \gamma_i^2 = \int f^2 \, dx < \infty, \quad \sum_{i=1}^\infty \gamma_i v_i = f \quad (11)
\]

**Proof.** a) Let \( f \in D^1 \). By Eqs. (7), (8), and Eqs. (10), with \( u \) and \( c_1 \) replaced by \( f \) and \( \gamma_1 \), we have
\[
\int_G \nabla \left( f - \sum_{i=1}^m \gamma_i v_i \right)^2 \, dx = \int_G f^2 \, dx - \sum_{i=1}^m \lambda_i^2 \gamma_i^2 \quad (12)
\]

Therefore, the series \( \Sigma \lambda_i^2 \gamma_i^2 \) converges. b) If it converges, then the series \( \Sigma \gamma_i^2 \) also converges (since \( \lambda_i \to +\infty \)), and we then have from Eqs. (7),
\[
\int_G \nabla \sum_{i=k}^m \gamma_i v_i \, dx = \sum_{i=k}^m \lambda_i \gamma_i v_i \, dx = \sum_{i=k}^m \gamma_i v_i \, dx = \sum_{i=k}^m \gamma_i v_i \, dx \leq \varepsilon
\]

for \( k, m > k \) (\( \varepsilon \)). By the Cauchy condition, the series \( \Sigma \gamma_i v_i \) converges in \( W^1_2 \) and in \( L_2 \). c) In \( L_2 \) it converges to \( f \), and hence it converges to \( f \) in \( W^1_2 \) and \( f \in W^1_2 \). Since \( v_i \in D^1 \), then \( f \in D^1 \) (see [31], p. 38).

**Theorem 1.** a) In order that \( f(x) \in D^k \), it is necessary and sufficient that \( f \in L^1 \) and that the following series be convergent:
\[
\sum_{i=1}^\infty \lambda_i^2 \gamma_i^2 \quad (13)
\]